

LARGE TIME BEHAVIOR OF SOLUTIONS TO 3-D MHD SYSTEM WITH INITIAL DATA NEAR EQUILIBRIUM

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ABSTRACT. In [7], Califano and Chiuderi conjectured that the energy of incompressible Magnetic hydrodynamical system is dissipated at a rate that is independent of the ohmic resistivity. The goal of this paper is to mathematically justify this conjecture in three space dimension provided that the initial magnetic field and velocity is a small perturbation of the equilibrium state $(e_3, 0)$. In particular, we prove that for such data, 3-D incompressible MHD system without magnetic diffusion has a unique global solution. Furthermore, the velocity field and the difference between the magnetic field and e_3 decay to zero in both L^∞ and L^2 norms with explicit rates. We point out that the decay rate in the L^2 norm is optimal in sense that this rate coincides with that of the linear system. The main idea of the proof is to exploit Hörmander's version of Nash-Moser iteration scheme, which is very much motivated by the seminar papers [18, 19, 20] by Klainerman on the long time behavior to the evolution equations.

Keywords: MHD system, Nash-Moser iteration scheme, Littlewood-Paley theory, Besov spaces.

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1. INTRODUCTION

In this paper, we investigate the large time behavior of the global smooth solutions to the following three-dimensional incompressible magnetic hydrodynamical (or MHD in short) system with initial data being sufficiently close to the equilibrium state $(e_3, 0)$:

$$(1.1) \quad \begin{cases} \partial_t b + u \cdot \nabla b = b \cdot \nabla u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = b \cdot \nabla b, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ (b, u)|_{t=0} = (b_0, u_0) \quad \text{with} \quad b_0 = e_3 + \varepsilon \phi, \end{cases}$$

where $b = (b^1, b^2, b^3)$ denotes the magnetic field, $u = (u^1, u^2, u^3)$ and p stand for the velocity and scalar pressure of the fluid respectively. This MHD system (1.1) with zero diffusivity in the magnetic field equation can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small. One may check the references [5, 13, 14, 22] for more explanations to this system.

Whether there is dissipation or not for the magnetic field of (1.1) is a very important problem from physics of plasmas. The heating of high temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at the length scales which are much larger than the dissipative ones. It has been conjectured that in the two-dimensional MHD system, energy is dissipated at a rate that is independent of the ohmic resistivity [7]. In other words, the diffusivity for the magnetic field equation can be zero yet the whole system may still be dissipative. The goal of this paper is to rigorously justify this conjecture in three space dimension provided that the initial data of (1.1) is a small perturbation of the equilibrium state $(e_3, 0)$.

Concerning the well-posedness issue of the system (1.1), Chemin et al [12] proved the local well-posedness of (1.1) with initial data in the critical Besov spaces. Lin and the second author [24]

proved the global well-posedness to a modified three-dimensional MHD system with initial data sufficiently close to the equilibrium state (see [25] for a simplified proof). Lin, Xu and the second author [23] established the global well-posedness of (1.1) in 2-D provided that the initial data is near the equilibrium state $(e_d, 0)$ and the initial magnetic field, b_0 , satisfies sort of admissible condition, namely

$$(1.2) \quad \int_{\mathbb{R}} (b_0 - e_3)(Z(t, \alpha)) dt = 0 \quad \text{for all} \quad \alpha \in \mathbb{R}^d \times \{0\}$$

with $Z(t, \alpha)$ being determined by

$$\begin{cases} \frac{dZ(t, \alpha)}{dt} = b_0(Z(t, \alpha)), \\ Z(t, \alpha)|_{t=0} = \alpha. \end{cases}$$

Similar result in three space dimension was proved by Xu and the second author in [30].

In the 2-D case, the restriction (1.2) was removed by Ren, Wu, Xiang and Zhang in [27] by carefully exploiting the divergence structure of the velocity field. Moreover, the authors proved that

$$(1.3) \quad \|\partial_{x_2}^k b(t)\|_{L^2} + \|\partial_{x_2}^k u(t)\|_{L^2} \leq C \langle t \rangle^{-\frac{1+2k}{4} + \epsilon} \quad \text{for any } \epsilon \in]0, 1/2[\quad \text{and} \quad k = 0, 1, 2,$$

where $\langle t \rangle \stackrel{\text{def}}{=} (1 + t^2)^{\frac{1}{2}}$. A more elementary existence proof was also given by Zhang in [31]. Very recently, Abidi and the second author removed the restriction (1.2) in [1] for the 3-D MHD system. Moreover, if the initial magnetic field equals to e_3 and with other technical assumptions, this solution decays to zero according to

$$(1.4) \quad \|u(t)\|_{H^2} + \|b(t) - e_3\|_{H^2} \leq C \langle t \rangle^{-\frac{1}{4}}.$$

Note that (1.4) corresponding to the critical case of (1.3), that is, $\epsilon = 0$ in (1.3).

This idea of considering the global well-posedness of MHD system with initial data close to the equilibrium state $(e_d, 0)$ goes back to the work by Bardos, Sulem and Sulem [2] for the global well-posedness of ideal incompressible MHD system. In general, it is not known whether or not classical solutions of (1.1) can develop finite time singularities even in two dimension. In the case when there is full magnetic diffusion in (1.1), one may check [15] for its local well-posedness in the classical Sobolev spaces, and [28] for the global well-posedness of such a system in two space dimension. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu [8] (see also [9]) proved that such a system is globally well-posed for any data in $H^2(\mathbb{R}^2)$. Lately He, Xu and Yu [16] (see also [6] and [29]) justified the vanishing viscosity limit of the full diffusive MHD system to the solution constructed by Bardos et al in [2] for the ideal MHD system.

The main result of this paper states as follows:

Theorem 1.1. *Let $e_3 = (0, 0, 1)$, $b_0 = e_3 + \varepsilon \phi$ with $\phi = (\phi_1, \phi_2, \phi_3) \in C_c^\infty$ and $\text{div } \phi = 0$, let $u_0 \in W^{N_0, 1} \cap H^{N_0}$ for some integer N_0 sufficiently large. Then there exist sufficiently small positive constants ε_0, c_0 such that if*

$$(1.5) \quad \|u_0\|_{W^{N_0, 1}} + \|u_0\|_{H^{N_0}} \leq c_0 \quad \text{and} \quad \varepsilon \leq \varepsilon_0,$$

(1.1) has a unique global solution (b, u) so that for any $T > 0$, $b \in C^2([0, T] \times \mathbb{R}^3)$, $u \in C^2([0, T] \times \mathbb{R}^3)$. Moreover, for some $\kappa > 0$, there hold

$$(1.6) \quad \begin{aligned} \|u(t)\|_{W^{2, \infty}} &\leq C_\kappa \langle t \rangle^{-\frac{5}{4} + \kappa}, & \|b(t) - e_3\|_{W^{2, \infty}} &\leq C_\kappa \langle t \rangle^{-\frac{3}{4} + \kappa} \quad \text{and} \\ \|u(t)\|_{H^2} + \|b(t) - e_3\|_{H^2} &\leq C \langle t \rangle^{-\frac{1}{2}}, & \|\nabla u(t)\|_{L^2} &\leq C \langle t \rangle^{-1}. \end{aligned}$$

Let us remark that the above theorem recovers the global well-posedness result of the system (1.1) in [1]. Moreover, the bigger the integer N_0 , the smaller the positive constant κ . The main idea of the proof here works in both two space dimension and in three space dimension. The L^∞ decay rates of the solution in (1.6) are completely new. The L^2 decay rates of the solution are optimal in the sense that these decay rates coincide with those of the linearized system (see Propositions 2.1 and 2.5 below), which greatly improves the rate given by (1.4). We can also work on the decay rates for the higher order derivatives of the solutions. But we choose not to pursue on this direction here.

2. STRUCTURE AND STRATEGIES OF THE PROOF

2.1. Lagrangian formulation of (1.1). As observed in the previous references ([23, 30]), the linearized system of (1.1) around the equilibrium state $(e_3, 0)$ reads

$$(2.1) \quad \begin{cases} Y_{tt} - \Delta Y_t - \partial_3^2 Y = \mathbf{f} & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ Y|_{t=0} = Y^{(0)}, \quad Y_t|_{t=0} = Y^{(1)}. \end{cases}$$

It is easy to calculate that this system has two different eigenvalues

$$(2.2) \quad \lambda_1(\xi) = -\frac{|\xi|^2}{2} + \sqrt{\frac{|\xi|^4}{4} - \xi_3^2} \quad \text{and} \quad \lambda_2(\xi) = -\frac{|\xi|^2}{2} - \sqrt{\frac{|\xi|^4}{4} - \xi_3^2}.$$

The Fourier modes corresponding to $\lambda_2(\xi)$ decays like $e^{-t|\xi|^2}$. Whereas the decay property of the Fourier modes corresponding to $\lambda_-(\xi)$ varies with directions of ξ as

$$\lambda_1(\xi) = -\frac{2\xi_3^2}{|\xi|^2(1 + \sqrt{1 - \frac{4\xi_3^2}{|\xi|^4}})} \rightarrow -1 \quad \text{as } |\xi| \rightarrow \infty$$

only in the ξ_3 direction. This simple analysis shows that the dissipative properties of the system (2.1) may be more complicated than that for the linearized system of isentropic compressible Navier-Stokes system (see [11] for instance). Moreover, it is well-known that it is in general impossible to propagate the anisotropic regularities for the transport equation. This motivates us to use the Lagrangian formulation of the system (1.1).

Let us now recall the Lagrangian formulation of (1.1) from [1]. Let (b, u) be a smooth enough solution of (1.1), we define

$$(2.3) \quad \begin{aligned} X(t, y) &= y + \int_0^t u(t', X(t', y)) dt' \stackrel{\text{def}}{=} y + Y(t, y), \quad \mathbf{u}(t, y) \stackrel{\text{def}}{=} u(t, X(t, y)), \\ \mathbf{b}(t, y) &\stackrel{\text{def}}{=} b(t, X(t, y)), \quad \mathbf{p}(t, y) \stackrel{\text{def}}{=} p(t, X(t, y)), \quad \mathcal{A} \stackrel{\text{def}}{=} (Id + \nabla_y Y)^{-1} \quad \text{and} \quad \nabla_Y \stackrel{\text{def}}{=} \mathcal{A}^t \nabla_y. \end{aligned}$$

Then $(Y, \mathbf{b}, \mathbf{u}, \mathbf{p})$ solves

$$(2.4) \quad \begin{cases} \mathbf{b}(t, y) = \partial_{b_0} X(t, y), \quad \nabla_Y \cdot \mathbf{b} = 0, \\ Y_{tt} - \Delta_y Y_t - \partial_{b_0}^2 Y = \partial_{b_0} b_0 + g, \\ Y|_{t=0} = Y^{(0)} = 0, \quad Y_t|_{t=0} = Y^{(1)} = u_0(y), \end{cases}$$

where

$$(2.5) \quad \begin{aligned} g &= \text{div}_y [(\mathcal{A} \mathcal{A}^t - Id) \nabla_y Y_t] - \mathcal{A}^t \nabla_y \mathbf{p}, \quad \partial_{b_0} \stackrel{\text{def}}{=} b_0 \cdot \nabla_y, \quad \text{and} \\ (\Delta_x p)(t, X(t, y)) &= \sum_{i,j=1}^3 \nabla_{Y^i} \nabla_{Y^j} (\partial_{b_0} X^i \partial_{b_0} X^j - Y_t^i Y_t^j)(t, y). \end{aligned}$$

In what follows, we assume that

$$(2.6) \quad \text{supp}(b_0(x_h, \cdot) - e_3) \subset [0, K] \quad \text{and} \quad b_0^3 \neq 0.$$

Due to the difficulty of the variable coefficients for the linearized system of (2.4), we shall use Frobenius Theorem type argument to find a new coordinate system $\{z\}$ so that $\partial_{b_0} = \partial_{z_3}$. Toward this, let us define

$$(2.7) \quad \begin{cases} \frac{dy_1}{dy_3} = \frac{b_0^1}{b_0^3}(y_1, y_2, y_3), & y_1|_{y_3=0} = w_1, \\ \frac{dy_2}{dy_3} = \frac{b_0^2}{b_0^3}(y_1, y_2, y_3), & y_2|_{y_3=0} = w_2, \\ y_3 = w_3, \end{cases}$$

and

$$(2.8) \quad z_1 = w_1, \quad z_2 = w_2, \quad z_3 = w_3 + \int_0^{w_3} \left(\frac{1}{b_0^3(y(w))} - 1 \right) dw'_3.$$

Then we have

$$(2.9) \quad \partial_{b_0} f(y) = \frac{\partial f(y(w(z)))}{\partial z_3}, \quad \text{and} \quad \nabla_y = \nabla_Z = \mathcal{B}^t(z) \nabla_z \quad \text{with} \quad \mathcal{B}(z) = \left(\frac{\partial y(w(z))}{\partial z} \right)^{-1}.$$

It is easy to observe that

$$\begin{aligned} \mathcal{B}(z) &= \left(\frac{\partial y(w(z))}{\partial z} \right)^{-1} = \left(\frac{\partial y(w(z))}{\partial w} \times \frac{\partial w(z)}{\partial z} \right)^{-1} \\ &= \left(\frac{\partial w(z)}{\partial z} \right)^{-1} \left(\frac{\partial y(w(z))}{\partial w} \right)^{-1} = \left(\frac{\partial z}{\partial w} \right) \left(\frac{\partial y(w(z))}{\partial w} \right)^{-1}. \end{aligned}$$

Yet it follows from (2.7) that

$$(2.10) \quad \begin{aligned} \left(\frac{\partial y(w)}{\partial w} \right) &= \begin{pmatrix} 1 & 0 & \frac{b_0^1}{b_0^3} \\ 0 & 1 & \frac{b_0^2}{b_0^3} \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \int_0^{w_3} \frac{\partial}{\partial y_1} \left(\frac{b_0^1}{b_0^3} \right) dy'_3 & \int_0^{w_3} \frac{\partial}{\partial y_2} \left(\frac{b_0^1}{b_0^3} \right) dy'_3 & 0 \\ \int_0^{w_3} \frac{\partial}{\partial y_1} \left(\frac{b_0^2}{b_0^3} \right) dy'_3 & \int_0^{w_3} \frac{\partial}{\partial y_2} \left(\frac{b_0^2}{b_0^3} \right) dy'_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial w_1} & \frac{\partial y_1}{\partial w_2} & \frac{\partial y_1}{\partial w_3} \\ \frac{\partial y_2}{\partial w_1} & \frac{\partial y_2}{\partial w_2} & \frac{\partial y_2}{\partial w_3} \\ \frac{\partial y_3}{\partial w_1} & \frac{\partial y_3}{\partial w_2} & \frac{\partial y_3}{\partial w_3} \end{pmatrix} \\ &\stackrel{\text{def}}{=} A_1(y(w)) + A_2(y(w)) \left(\frac{\partial y(w)}{\partial w} \right), \end{aligned}$$

which gives

$$(2.11) \quad \left(\frac{\partial y(w)}{\partial w} \right) = (Id - A_2(y(w)))^{-1} A_1(y(w)).$$

While it is easy to observe that

$$(2.12) \quad \left(\frac{\partial z(w)}{\partial w} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \int_0^{w_3} \frac{\partial}{\partial w_1} \left(\frac{1}{b_0^3(y(w))} \right) dw'_3 & \int_0^{w_3} \frac{\partial}{\partial w_2} \left(\frac{1}{b_0^3(y(w))} \right) dw'_3 & \frac{1}{b_0^3} \end{pmatrix} \stackrel{\text{def}}{=} A_3(w).$$

As a consequence, we obtain

$$(2.13) \quad \begin{aligned} y(w) &= (y_h(w_h, w_3), w_3), \quad w(z) = (z_h, w_3(z)), \quad \text{and} \quad y(w(z)) = (y_h(z_h, w_3(z)), w_3(z)), \\ \mathcal{B}(z) &= A_3(w(z)) A_1^{-1}(y(w(z))) (Id - A_2(w(z))), \end{aligned}$$

with the matrices A_1, A_2, A_3 being determined by (2.10) and (2.12) respectively.

For simplicity, let us abuse the notation that $Y(t, z) = Y(t, y(w(z)))$. Then the system (2.4) becomes

$$(2.14) \quad \begin{cases} Y_{tt} - \Delta_z Y_t - \partial_{z_3}^2 Y = (\nabla_Z \cdot \nabla_Z - \Delta_z) Y_t + \partial_{z_3} b_0(y(w(z))) + g(y(w(z))), \\ Y|_{t=0} = Y_0 = 0, \quad Y_t|_{t=0} = Y_1(z) = u_0(y(w(z))), \end{cases}$$

for g given by (2.4). Since $\partial_{z_3} b_0(y(w(z)))$ in the source term is a time independent function, we now introduce a smooth cut-off function $\eta(z_3)$ with $\eta(z_3) = \begin{cases} 0, & z_3 \geq 2 + K, \\ 1, & -1 \leq z_3 \leq 1 + K, \\ 0, & z_3 \leq -2, \end{cases}$ and a correction term \tilde{Y} so that $Y = \tilde{Y} + \bar{Y}$ and

$$(2.15) \quad \tilde{Y}(z) = \eta(z_3) \left(\int_{-1}^{z_3} (e_3 - b_0(y(w(z_h, z'_3)))) dz'_3 - \int_{-1}^{K+1} (e_3 - b_0(y(w(z_h, z'_3)))) dz'_3 \right),$$

which satisfies

$$(2.16) \quad \partial_{z_3} \tilde{Y}(z) = e_3 - b_0(y(w(z))), \quad \text{and} \quad \partial_{z_3} (\partial_{z_3} \tilde{Y} + b_0(y(w(z)))) = 0.$$

Then in view of (2.23), (2.24) and (2.30) of [1], \bar{Y} solves

$$(2.17) \quad \begin{cases} \bar{Y}_{tt} - \Delta_z \bar{Y}_t - \partial_{z_3}^2 \bar{Y} = f, \\ \bar{Y}|_{t=0} = \bar{Y}^{(0)} = -\tilde{Y}, \quad \bar{Y}_{t=0} = Y^{(1)}, \end{cases}$$

with

$$(2.18) \quad \begin{aligned} \mathcal{A} &= \left(Id + \mathcal{B}^t \nabla_z \tilde{Y} + \mathcal{B}^t \nabla_z \bar{Y} \right)^{-1}, \quad \text{and} \\ f &= \mathcal{B}^t \nabla_z \cdot [(\mathcal{A} \mathcal{A}^t - Id) \mathcal{B}^t \nabla_z \bar{Y}_t] + \mathcal{B}^t \nabla_z \cdot (\mathcal{B}^t \nabla_z \bar{Y}_t) - \Delta_z \bar{Y}_t - (\mathcal{B} \mathcal{A})^t \nabla_z \mathbf{p}, \\ \nabla_z \mathbf{p} &= -\nabla_z \Delta_z^{-1} \operatorname{div}_z (\det(\mathcal{B}^{-1}) (\mathcal{B} \mathcal{A} \mathcal{A}^t \mathcal{B}^t - Id) \nabla_z \mathbf{p}) \\ &\quad - \nabla_z \Delta_z^{-1} \operatorname{div}_z ((\det(\mathcal{B}^{-1}) Id - Id) \nabla_z \mathbf{p}) \\ &\quad + \nabla_z \Delta_z^{-1} \operatorname{div}_z \left(\mathcal{B} \mathcal{A} \operatorname{div}_z (\det(\mathcal{B}^{-1}) \mathcal{B} \mathcal{A} (\partial_3 \bar{Y} \otimes \partial_3 \bar{Y} - \bar{Y}_t \otimes \bar{Y}_t)) \right). \end{aligned}$$

2.2. The proof of Theorem 1.1. Before presenting the main result for the system (2.17-2.18), let us first introduce notations of the norms: For $f : \mathbb{R}_y^3 \rightarrow \mathbb{R}$, $u : \mathbb{R}^+ \times \mathbb{R}_y^3 \rightarrow \mathbb{R}$, and $p \in [1, +\infty]$, $N \in \mathbb{N}$, we denote

$$\|f\|_{W^{N,p}} \stackrel{\text{def}}{=} \sum_{|\alpha| \leq N} \|D_y^\alpha f\|_{L^p} \quad \text{and} \quad \|u\|_{L^p; k, N} \stackrel{\text{def}}{=} \sup_{t>0} (1+t)^k \|u(t)\|_{W^{N,p}}.$$

In particular, when $p = 1$, $p = 2$ and $p = \infty$, we simplify the notations as

$$(2.19) \quad \begin{aligned} \|f\|_N &\stackrel{\text{def}}{=} \|f\|_{W^{N,1}}, \quad \|f\|_N \stackrel{\text{def}}{=} \|f\|_{H^N}, \quad |f|_N \stackrel{\text{def}}{=} \|f\|_{W^{N,\infty}} \\ \text{and} \quad \|u\|_{k,N} &\stackrel{\text{def}}{=} \|u\|_{L^2; k, N}, \quad |u|_{k,N} \stackrel{\text{def}}{=} \|u\|_{L^\infty; k, N}. \end{aligned}$$

Theorem 2.1. *There exist an integer L_0 and small constants $\eta, \varepsilon_0 > 0$ such that if*

$$(2.20) \quad \|(\bar{Y}^{(0)}, Y^{(1)})\|_{L_0} + \|(\bar{Y}^{(0)}, Y^{(1)})\|_{L_0} \leq \eta \quad \text{and} \quad \varepsilon \leq \varepsilon_0.$$

Then the system (2.17) has a unique global solution $\bar{Y} \in C^2([0, \infty); C^{N_1-4}(\mathbb{R}^3))$, where $N_1 = [(L_0 - 12)/2]$. Furthermore, for any $\kappa > 0$, there hold

$$(2.21) \quad |\partial_3 \bar{Y}|_{\frac{3}{4}-\kappa, 2} + |\bar{Y}_t|_{\frac{5}{4}-\kappa, 2} + |\bar{Y}|_{\frac{1}{4}-\kappa, 2} \leq C_\kappa \eta,$$

and

$$(2.22) \quad \begin{aligned} &\| |D|^{-1} (\partial_3 \bar{Y}, \bar{Y}_t) \|_{0, N_1+2} + \|\nabla \bar{Y}\|_{0, N_1+1} + \|(\bar{Y}_t, \partial_3 \bar{Y})\|_{\frac{1}{2}, N_1+1} + \|\nabla \bar{Y}_t\|_{1, N_1-1} \\ &\quad + \|\bar{Y}_t\|_{L_t^2(H^{N_1+2})} + \|(\partial_3 \bar{Y}, \langle t \rangle^{\frac{1}{2}} \nabla \bar{Y}_t)\|_{L_t^2(H^{N_1+1})} + \|\bar{Y}_{tt}\|_{\frac{1}{2}, N_1-2} \leq C. \end{aligned}$$

Admitting Theorem 2.1 for the time being, let us now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Indeed, in view of (2.3), one has

$$(2.23) \quad \begin{aligned} Y_1(z) &= u_0(y_h(z_h, w_3(z)), w_3(z)) \quad \text{and} \quad \mathbf{u}(t, y) = Y_t(t, y + Y(t, y)), \\ \mathbf{b}(t, y) &= b_0(y) + b_0(y) \cdot \nabla_y Y(t, y) \quad \text{with} \quad Y(t, (y_h(z_h, w_3(z)), w_3(z))) = \tilde{Y}(z) + \bar{Y}(t, z), \end{aligned}$$

with $\tilde{Y}(z)$ and $\bar{Y}(t, z)$ being determined by (2.15) and (2.17) respectively.

Whereas in view of (2.10), (2.12) and (2.13), we get, by a similar proof of Lemma 4.3 of [1] that for any $N \in \mathbb{N}$,

$$(2.24) \quad |(\mathcal{B} - Id)|_N \leq C_N \varepsilon.$$

So that under the assumptions of (1.5), there holds (2.20). Then Theorem 2.1 ensures that the system (2.17-2.18) has a unique global classical solution $\bar{Y} \in C^2([0, \infty); C^{N_1-4}(\mathbb{R}^3))$, which verifies (2.21) and (2.22). In particular, it follows from (2.15) and (2.21) that

$$|\nabla_z Y|_{0,1} \leq |\partial_z \tilde{Y}|_1 + |\nabla \bar{Y}|_{0,1} \leq C(\varepsilon + \eta),$$

which together with (2.23) ensures that $\mathbf{u} \in C^2([0, \infty) \times \mathbb{R}^3)$ and $\mathbf{b} \in C^2([0, \infty) \times \mathbb{R}^3)$. Furthermore due to

$$\left| \frac{\partial X}{\partial y} - Id \right|_{0,1} = |{}^t \mathcal{B} \nabla_z Y|_{0,1} \leq C(\varepsilon + \eta),$$

we deduce from (2.3) that $u \in C^2([0, \infty) \times \mathbb{R}^3)$ and $b \in C^2([0, \infty) \times \mathbb{R}^3)$ which verifies the system (1.1) thanks to the derivation at the beginning of Subsection 2.1.

On the other hand, by virtue of (2.16), we have

$$\mathbf{b}(t, y(w(z))) = b_0(y(w(z))) + \partial_3 \tilde{Y}(z) + \partial_3 \bar{Y}(t, z) = e_3 + \partial_3 \bar{Y}(t, z),$$

which together with (2.21), (2.22) and (2.23) implies that there holds (1.6). This completes the proof of Theorem 1.1. \square

2.3. Strategies of the proof to Theorem 2.1. Observing from the calculations in [1] that under the assumptions of Theorem 1.1, the matrix \mathcal{B} given by (2.13) is sufficiently close to the identity matrix in the norms of $W^{N_0,1}$ and H^{N_0} as long as ε is sufficiently small. To avoid cumbersome calculation, here we just prove Theorem 2.1 for the system (2.1) with

$$(2.25) \quad \begin{aligned} \mathcal{A} &= (Id + \nabla_y Y)^{-1}, \quad f = \nabla_y \cdot ((\mathcal{A} \mathcal{A}^t - Id) \nabla_y Y_t) - \mathcal{A}^t \nabla_y \mathbf{p}, \quad \text{and} \\ \mathbf{p} &= -\Delta_y^{-1} \operatorname{div}_y ((\mathcal{A} \mathcal{A}^t - Id) \nabla_y \mathbf{p}) + \Delta_y^{-1} \operatorname{div}_y (\mathcal{A} \operatorname{div}_y (\mathcal{A} (\partial_{y_3} Y \otimes \partial_{y_3} Y - Y_t \otimes Y_t))), \end{aligned}$$

which corresponds to $\mathcal{B} = Id$ in (2.17). The general case follows along the same line.

Let us remark that the system (2.1) is not scaling, rotation and Lorentz invariant, so that Klainerman's vector field method ([21]) can not be applied here. Yet the ideas developed by Klainerman in the seminar papers [18, 19, 20] can be well adapted for this system. We now recall the classical result on the global well-posedness to some evolutionary system from [19]. Let us consider the following system

$$(2.26) \quad \begin{cases} u_t - \mathcal{L}u = F(u, Du) & \text{with} \quad Du = (u_t, u_{x_1}, \dots, u_{x_d}), \\ u|_{t=0} = u, & Pu_0 = 0, \end{cases}$$

where $\mathcal{L} \stackrel{\text{def}}{=} \sum_{|\alpha| \leq \gamma} a_\alpha D_x^\alpha$ with a_α being $r \times r$ matrices with constant entries. Under the assumptions that

- (1) \mathcal{L} satisfies a dissipative condition of the following type: there exists a positive definite $r \times r$ matrix A such that

$$\text{either } \int_{\mathbb{R}^d} \Re(A \mathcal{L} f, f) dx \leq 0 \quad \text{or} \quad \int_{\mathbb{R}^d} \Re(A \mathcal{L} f, f) dx \leq -\|\nabla f\|_{L^2}^2$$

for any $f \in C_c^\infty$;

(2) Let $\Gamma(t)u_0$ be the solution of

$$\partial_t u - \mathcal{L}u = 0 \quad \text{and} \quad u(0, x) = u_0(x).$$

There is a differential matrix P such that

$$|\Gamma(t)u_0|_0 \leq C\langle t \rangle^{-k_0} \|u_0\|_d$$

for any $u_0 \in W^{d,1} \cap L^\infty$ that satisfies $Pu_0 = 0$;

(3) $AF_{u_t}, AF_{u_{x_i}}, i = 1, \dots, d$, are symmetric matrices and F_{u_t} is independent of u_t . Moreover

$$(2.27) \quad |F(u, Du)| \leq C(|u| + |Du|)^{p+1} \quad \text{for } |u| + |Du| \text{ sufficiently small};$$

(4) p is an integer and F is a smooth function so that there holds

$$(2.28) \quad \frac{1}{p} \left(1 + \frac{1}{p} \right) < k_0;$$

Klainerman proved in [19] the following celebrated theorem:

Theorem 2.2 (Theorem 1 of [19]). *There exist an integer $N_0 > 0$ and a small constant $\eta > 0$ such that if*

$$\|u_0\|_{N_0} + \|u_0\|_{N_0} \leq \eta,$$

(2.26) has a unique solution $u \in C^1([0, T]; C^\gamma)$ for any $T > 0$. Moreover, the solution behaves, for t large, like

$$(2.29) \quad |u(t, x)| = O\left(t^{-\frac{1+\varepsilon}{p}}\right) \quad \text{as } t \rightarrow \infty,$$

for some small $\varepsilon > 0$. Also

$$(2.30) \quad \|u(t)\|_{L^2} = O(1) \quad \text{as } t \rightarrow \infty.$$

Let us remark that due to the appearance of the double Riesz transform in the expression of f in (2.25), the source term f in (2.1) can not satisfy the growth condition (2.27); secondly, even if we can assume the source term f is in quadratic growth of $(Y_t, \partial_3 Y)$, that corresponds to $p = 1$ in (2.27), the growth rate obtained in (3.2) below does not meet the requirement of (2.28). This makes it impossible to apply Theorem 2.2 for the system (2.1). Yet by considering the specific anisotropic structure of the system (2.1), we can still succeed in applying Nash-Moser scheme to establish the global existence as well as the large time behavior of solutions to (2.1-2.25).

Now we outline the proof of Theorem 2.1. According to the strategy in [18, 19, 20], the first step is to study the decay properties of the linear system:

$$(2.31) \quad \begin{cases} \mathcal{Y}_{tt} - \Delta \mathcal{Y}_t - \partial_3^2 \mathcal{Y} = 0, \\ \mathcal{Y}|_{t=0} = \mathcal{Y}_0, \quad \mathcal{Y}_t|_{t=0} = \mathcal{Y}_1. \end{cases}$$

Proposition 2.1. *Let $\mathcal{Y}(t)$ be a smooth enough solution of (2.31). Given $\delta \in [0, 1]$, $N \in \mathbb{N}$, there exist $C_{\delta, N}, C_N > 0$ such that*

$$(2.32) \quad |\partial_3 \mathcal{Y}|_{1, N} + |\partial_t \mathcal{Y}|_{\frac{3}{2}-\delta, N} + |\mathcal{Y}|_{\frac{1}{2}, N} \leq C_{\delta, N} (\| |D|^{2\delta} (\mathcal{Y}_0, \mathcal{Y}_1) \|_{L^1} + \| |D|^{N+4} (\Delta \mathcal{Y}_0, \mathcal{Y}_1) \|_{L^1});$$

$$(2.33) \quad \begin{aligned} & \|(\partial_t \mathcal{Y}, \partial_3 \mathcal{Y})\|_{L_t^\infty(H^{N+1})} + \|\Delta \mathcal{Y}\|_{L_t^\infty(H^N)} + \|\nabla \partial_t \mathcal{Y}\|_{L_t^2(H^{N+1})} \\ & + \|\nabla \partial_3 \mathcal{Y}\|_{L_t^2(H^N)} \leq C_N (\|(\partial_3 \mathcal{Y}_0, \mathcal{Y}_1)\|_{N+1} + \|\Delta \mathcal{Y}_0\|_N); \end{aligned}$$

$$(2.34) \quad \|\langle t \rangle^{\frac{1}{2}} (\partial_t \mathcal{Y}, \partial_3 \mathcal{Y})\|_{L_t^\infty(H^N)} + \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t \mathcal{Y}\|_{L_t^2(H^N)} \leq C_N (\| |D|^{-1} (\partial_3 \mathcal{Y}_0, \mathcal{Y}_1) \|_{N+1} + \|\nabla \mathcal{Y}_0\|_N);$$

$$(2.35) \quad \|\langle t \rangle \Delta \partial_t \mathcal{Y}\|_{L_t^\infty(H^N)} \leq C_N \|(\Delta \mathcal{Y}_0, \mathcal{Y}_1)\|_{N+2}.$$

We emphasize here the estimates of (2.32) and (2.33) are of anisotropic type, which means that the decay rates of the partial derivatives of the solution to (2.31) are different, which is consistent with the heuristic discussions at the beginning of Section 2. Moreover, the estimate of (2.32) is valid for $\delta = 0$. Similar estimates as (2.34) and (2.35) were not proved in [18, 19, 20]. They are purely due to the special structure of the linearized system (2.31).

With the above proposition, we next turn to the decay estimates for the solutions of the following inhomogeneous equation of (2.31)

$$(2.36) \quad \begin{cases} Y_{tt} - \Delta Y_t - \partial_3^2 Y = g, \\ Y|_{t=0} = Y_t|_{t=0} = 0. \end{cases}$$

Proposition 2.2. *Let $\delta \in [0, 1/4[$ and $\theta \in [1, \infty[$. We assume that $g(t) = 0$ if $t \geq \theta$. Then the solution Y to (2.36) verifies for any $N \geq 0$,*

$$(2.37) \quad |\partial_3 Y|_{1,N} + |\partial_t Y|_{\frac{3}{2}-\delta,N} + |Y|_{\frac{1}{2},N} \leq C_{\delta,N} R_{N,\theta}(g),$$

where

$$(2.38) \quad R_{N,\theta}(g) \stackrel{\text{def}}{=} \|g\|_{L_t^1(\delta,N)} + \theta^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1} g\|_{L_t^2(H^{N+3})} + \log(\theta) \| |D|^{-1} g \|_{\frac{3}{2}-\delta,N+3},$$

where

$$(2.39) \quad \|g\|_{\delta,N} \stackrel{\text{def}}{=} \| |D|^{2\delta} g \|_{L^1} + \| |D|^{N+4} g \|_{L^1} \quad \text{and} \quad \|g\|_{L_t^p(\delta,N)} = \left(\int_0^t \|g(t')\|_{\delta,N}^p dt' \right)^{\frac{1}{p}}.$$

The proof of the above propositions will be presented in Section 3.

The goal of Section 4 is to calculate the linearized system of (2.1), which reads

$$(2.40) \quad \begin{cases} X_{tt} - \Delta X_t - \partial_3^2 X = f'(Y; X) + g, \\ X|_{t=0} = X_t|_{t=0} = 0, \end{cases}$$

where $f'(Y; X) = f'_0(Y; X) + f'_1(Y; X) + f'_2(Y; X)$, and $f'_0(Y; X)$, $f'_1(Y; X)$ and $f'_2(Y; X)$ are determined respectively by (4.6) and (4.7). Furthermore, the second derivative of $f''(Y; X, W)$ will be presented in Subsection 4.2.

In Section 5, we shall derive the $\dot{W}^{2\delta,1} \cap \dot{W}^{N+4,1}$ and \dot{H}^{N+1} estimates for the source term $f'(Y; X)$ in the linearized system (2.40), which will be used to derive the decay estimates for the solutions of (2.40). The main result reads

Proposition 2.3. *Let the functionals, $f'_0(Y; X)$, $f'_1(Y; X)$, $f'_2(Y; X)$, be given by (4.6) and (4.7) respectively, and the norm $\|\cdot\|_{\delta,N}$ be given by (2.39). Then under the assumptions that $\delta > 0$, and*

$$(2.41) \quad \|\nabla Y\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \leq \delta_1 \quad \text{and} \quad \|\nabla Y\|_{\dot{B}_{2,1}^{\frac{5}{2}}} \leq 1,$$

for some $\delta_1 > 0$ sufficiently small, we have

$$(2.42) \quad \begin{aligned} \|f'_0(Y; X)\|_{\delta,N} &\leq \|\nabla Y\|_0 \|\nabla X_t\|_{N+6} + \|\nabla Y\|_{N+6} \|\nabla X_t\|_0 \\ &\quad + \|\nabla Y_t\|_0 \|\nabla X\|_{N+6} + (\|\nabla Y_t\|_{N+6} + \|\nabla Y\|_{N+6} \|\nabla Y_t\|_0) \|\nabla X\|_0, \end{aligned}$$

and

$$(2.43) \quad \|f'_1(Y; X)\|_{\delta,N} \lesssim f_1(\partial_3 Y, \partial_3 X) \quad \text{and}$$

$$(2.44) \quad \|f'_2(Y; X)\|_{\delta,N} \lesssim f_1(Y_t, X_t),$$

where the function $f_1(\mathbf{x}, \mathbf{y})$ is given by

$$\begin{aligned} f_1(\mathbf{x}, \mathbf{y}) &\stackrel{\text{def}}{=} \|\mathbf{x}\|_0 (\|\mathbf{y}\|_{N+6} + |\mathbf{x}|_0 \|\nabla X\|_{N+6}) \\ &\quad + \|\mathbf{y}\|_1 (\|\mathbf{x}\|_{N+6} + \|\nabla Y\|_{N+6} |\mathbf{x}|_1) + (\|\mathbf{x}\|_{N+6} + \|\nabla Y\|_{N+6} \|\mathbf{x}\|_3) |\mathbf{x}|_1 \|\nabla X\|_1. \end{aligned}$$

Proposition 2.4. *Under the assumption of Proposition 2.3, we have*

$$(2.45) \quad \begin{aligned} \| |D|^{-1} f'_0(Y; X) \|_{N+1} &\lesssim \| \nabla Y \|_0 \| \nabla X_t \|_{N+1} + \| \nabla Y \|_{N+1} \| \nabla X_t \|_0 \\ &\quad + \| Y_t \|_1 \| \nabla X \|_{N+1} + (\| Y_t \|_{N+2} + \| Y_t \|_1 \| \nabla Y \|_{N+1}) \| \nabla X \|_0, \end{aligned}$$

and

$$(2.46) \quad \| |D|^{-1} f'_1(Y; X) \|_{N+1} \lesssim f_2(\partial_3 Y, \partial_3 X) \quad \text{and}$$

$$(2.47) \quad \| |D|^{-1} f'_2(Y; X) \|_{N+1} \lesssim f_2(Y_t, X_t),$$

where the functional $f_2(\mathfrak{x}, \mathfrak{y})$ is given by

$$\begin{aligned} f_2(\mathfrak{x}, \mathfrak{y}) &\stackrel{\text{def}}{=} (\| \mathfrak{x} \|_1^{\frac{4}{3}} \| \mathfrak{x} \|_0^{\frac{2}{3}} + \| \mathfrak{x} \|_1^2) (\| \nabla X \|_{N+1} + \| \nabla Y \|_{N+1} \| \nabla X \|_1) + \| \mathfrak{x} \|_0 \| \mathfrak{y} \|_{N+1} \\ &\quad + (\| \mathfrak{x} \|_{N+1} + \| \nabla Y \|_{N+1} \| \mathfrak{x} \|_1) \| \mathfrak{y} \|_1 + (\| \mathfrak{x} \|_0^{\frac{1}{3}} \| \mathfrak{x} \|_0^{\frac{2}{3}} + \| \mathfrak{x} \|_0) \| \mathfrak{x} \|_{N+1} \| \nabla X \|_1. \end{aligned}$$

Let us remark that Riesz transform does not map continuously from L^1 to L^1 . Nevertheless due to (4.8) and (4.9), we can not avoid estimates of this type. To overcome this difficulty, a natural replacement of $\dot{W}^{s,1}$ will be the Besov space $\dot{B}_{1,1}^s$, which satisfies

$$\| \nabla (-\Delta)^{-\frac{1}{2}} |D|^s(f) \|_{L^1} \lesssim \| f \|_{\dot{B}_{1,1}^s} \quad \forall s \in \mathbb{R}.$$

We now recall the precise definition of the Besov norms from [3] for instance.

Definition 2.1. *Let us consider a smooth function φ on \mathbb{R} , the support of which is included in $[3/4, 8/3]$ such that*

$$\forall \tau > 0, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \tau) = 1 \quad \text{and} \quad \chi(\tau) \stackrel{\text{def}}{=} 1 - \sum_{j \geq 0} \varphi(2^{-j} \tau) \in \mathcal{D}([0, 4/3]).$$

Let us define

$$\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j} |\xi|) \hat{a}), \quad \text{and} \quad S_j a = \mathcal{F}^{-1}(\chi(2^{-j} |\xi|) \hat{a}).$$

Let (p, r) be in $[1, +\infty]^2$ and s in \mathbb{R} . We define the Besov norm by

$$\| a \|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \| (2^{js} \| \Delta_j a \|_{L^p})_j \|_{\ell^r(\mathbb{Z})}.$$

We remark that in the special case when $p = r = 2$, the Besov spaces $\dot{B}_{p,r}^s$ coincides with the classical homogeneous Sobolev spaces \dot{H}^s . Moreover, we have the following product laws (see Corollary 2.54 of [3]):

$$(2.48) \quad \| ab \|_{\dot{B}_{p,r}^s} \leq C (\| a \|_{L^\infty} \| b \|_{\dot{B}_{p,r}^s} + \| a \|_{\dot{B}_{p,r}^s} \| b \|_{L^\infty}),$$

for $s > 0$, $(p, r) \in [1, +\infty]^2$. Due to the product law (2.48), we need the index δ to be positive in Proposition 2.3.

In Section 6, we investigate energy estimates for the solutions of the linearized equation (2.40).

Theorem 2.3. *Let Y be a smooth enough vector field and X be a smooth solution to the linearized equation (2.40). We assume that Y satisfies (2.41) and*

$$(2.49) \quad \| Y_t \|_{0,0} \leq 1, \quad \text{and} \quad \| Y_t \|_{0,1} \leq 1.$$

Then for any $\varepsilon > 0$, we have

$$(2.50) \quad \begin{aligned} \mathcal{E}_0(t) &\leq C_\varepsilon \| \langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g \|_{L_t^2(H^1)} E_\varepsilon(Y) \quad \text{and} \quad \text{for } N \geq 1 \\ \mathcal{E}_N(t) &\leq C_{\varepsilon,N} \left(\| \langle t \rangle^{\frac{1+\varepsilon}{2}} g \|_{L_t^2(H^N)} + \gamma_{\varepsilon,N+1}(Y) \| \langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g \|_{L_t^2(H^1)} \right) E_\varepsilon(Y), \end{aligned}$$

where

$$(2.51) \quad \begin{aligned} \mathcal{E}_N(t) &\stackrel{\text{def}}{=} \| |D|^{-1}(X_t, \partial_3 X) \|_{0,N+2} + \|\nabla X\|_{0,N+1} + \|X_t\|_{L_t^2(H^{N+2})} + \|\partial_3 X\|_{L_t^2(H^{N+1})}; \\ E_\varepsilon(Y) &\stackrel{\text{def}}{=} \exp\left(C(|\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^{\frac{4}{3}} \|\partial_3 Y\|_{L_t^2(L^2)}^{\frac{2}{3}} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^2 + |Y_t|_{1+\varepsilon,2})\right), \end{aligned}$$

and

$$(2.52) \quad \begin{aligned} \gamma_{\varepsilon,N+1}(Y) &\stackrel{\text{def}}{=} 1 + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,N+1} (1 + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}) + |Y_t|_{1+\varepsilon,N+2} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^{\frac{1}{3}} \|\partial_3 Y\|_{L_t^2(L^2)}^{\frac{2}{3}} \times \\ &\times (|\partial_3 Y|_{\frac{1}{2}+\varepsilon,N+1} + \|\nabla Y\|_{0,N+1}) + \|\nabla Y\|_{0,N+1} (1 + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,0}^2 + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,0} + |Y_t|_{1+\varepsilon,1}). \end{aligned}$$

We notice that when we perform the energy estimates for the derivatives of the solutions to (2.40), we are not able to treat the term $\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t)$, which appears in $f'_0(Y; X)$ (see (4.6)), as a source term. Instead, we need to rewrite (2.40) as

$$(2.53) \quad X_{tt} - \nabla \cdot \partial_t(\mathcal{A}\mathcal{A}^t \nabla X) - \partial_3^2 X = \tilde{f}'(Y; X) + g$$

where $\tilde{f}'(Y; X) = \tilde{f}'_0(Y; X) + f'_1(Y; X) + f'_2(Y; X)$ with $f'_m(Y; X), m = 1, 2$, given by (4.7), and $\tilde{f}'_0(Y; X)$ by

$$(2.54) \quad \tilde{f}'_0(Y; X) = -\nabla \cdot (\mathcal{A}(\nabla X \mathcal{A} + \mathcal{A}^t(\nabla X)^t)\mathcal{A}^t \nabla Y_t) - \nabla \cdot (\partial_t(\mathcal{A}\mathcal{A}^t)\nabla X).$$

With the energy estimates obtained in Theorem 2.3, we can work on the time-weighted energy estimate for the solutions of (2.40).

Corollary 2.1. *Under the assumptions of Theorem 2.3, we have*

$$(2.55) \quad \mathcal{E}_0 + \|(X_t, \partial_3 X)\|_{\frac{1}{2},1} + \|\langle t \rangle^{\frac{1}{2}} \nabla X_t\|_{L_t^2(H^1)} \leq C_\varepsilon \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L_t^2(H^1)} E_\varepsilon(Y),$$

and for $N \geq 1$,

$$(2.56) \quad \begin{aligned} \mathcal{E}_N + \|(X_t, \partial_3 X)\|_{\frac{1}{2},N+1} + \|\langle t \rangle^{\frac{1}{2}} \nabla X_t\|_{L_t^2(H^{N+1})} \\ \leq C_{\varepsilon,N} \left(\|\langle t \rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(H^N)} + \gamma_{\varepsilon,N+1}(Y) \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L_t^2(H^1)} \right) E_\varepsilon(Y). \end{aligned}$$

Proposition 2.5. *Under the assumptions of Theorem 2.3, we have for $N \geq 0$,*

$$(2.57) \quad \begin{aligned} \|\nabla X_t\|_{1,N} &\leq C_{\varepsilon,N} (\| |D|^{-1} g\|_{1+\varepsilon,N+2} + \|\nabla Y\|_{N+2} \| |D|^{-1} g\|_{1+\varepsilon,2}) \\ &+ C_{\varepsilon,N} \left(\|\langle t \rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(H^{N+1})} + \gamma_{\varepsilon,N+2}(Y) \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L_t^2(H^1)} \right. \\ &\quad \left. + \|\nabla Y\|_{0,N+2} (\|\langle t \rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(H^1)} + \gamma_{\varepsilon,2}(Y) \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L_t^2(H^1)}) \right) E_\varepsilon(Y). \end{aligned}$$

We emphasize that the decay estimates (2.57) can not be obtained by energy estimate. In fact, we will have to exploit anisotropic Littlewood-Paley analysis and the dissipative properties of the linear system (2.1). The proof of Proposition 2.5 will be presented in Section 7, which is of independent interest.

Let us summarize that under the assumptions (2.41), (2.49) and if we assume moreover

$$(2.58) \quad |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^{\frac{4}{3}} \|\partial_3 Y\|_{L_t^2(L^2)}^{\frac{2}{3}} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^2 + |Y_t|_{1+\varepsilon,2} \leq 1,$$

we have the following energy estimates: for $N \geq 0$, (we make the convention $\|u\|_{k,-1} = 0$)

$$(2.59) \quad \begin{aligned} \mathcal{E}_N + \|(X_t, \partial_3 X)\|_{\frac{1}{2},N+1} + \|\langle t \rangle^{\frac{1}{2}} \nabla X_t\|_{L_t^2(H^{N+1})} + \|\nabla X_t\|_{1,N-1} &\leq C_{\varepsilon,N} \left(\| |D|^{-1} g\|_{1+\varepsilon,N+1} \right. \\ &\quad \left. + \|\langle t \rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(H^N)} + \tilde{\gamma}_{\varepsilon,N+1}(Y) (\| |D|^{-1} g\|_{1+\varepsilon,2} + \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L_t^2(H^1)}) \right) \quad \text{with} \\ \tilde{\gamma}_{\varepsilon,N+1}(Y) &\leq C(1 + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,N+1} + |Y_t|_{1+\varepsilon,N+2} + \|\nabla Y\|_{0,N+1} + \|\nabla Y\|_{0,N+1}). \end{aligned}$$

In Section 8, we shall present the estimates to the nonlinear source term $f(Y)$ given by (2.25). The purpose of Section 9 is concerned with the related estimates for the second derivatives, $f''(Y; X, W)$, of the nonlinear functional $f(Y)$, computed in Section 4.2.

With the preparations in the previous sections, we can now exploit Nash-Moser iteration scheme to prove Theorem 2.1. In order to do so, we first recall some basic properties of the smoothing operator from [18, 19]. Let $\chi(t) \in C^\infty(\mathbb{R}; [0, 1])$ be such that

$$\chi(t) = 1 \text{ for } t \leq \frac{1}{2}, \quad \chi(t) = 0 \text{ for } t \geq 1.$$

Define for $\theta \geq 1$, the (cutoff-in-time) operator

$$(2.60) \quad S^{(1)}(\theta)Y(t, y) \stackrel{\text{def}}{=} \chi\left(\frac{t}{\theta}\right)Y(t, y).$$

Then we have

$$|S^{(1)}(\theta)Y|_{k,N} \leq C_{k,s}\theta^{k-s}|Y|_{s,N}, \quad \text{if } k \geq s \geq 0,$$

and

$$|(1 - S^{(1)}(\theta))Y|_{s,N} \leq C_{k,s}\theta^{-(k-s)}|Y|_{k,N} \quad \text{if } k \geq s \geq 0.$$

For $\theta' \geq 1$, we define the usual mollifying operator $S^{(2)}(\theta')$ in the space variables by

$$(2.61) \quad S^{(2)}(\theta')Y(t, y) \stackrel{\text{def}}{=} \widehat{\varphi}\left(\frac{D_y}{\theta'}\right)Y(t, y) = (\theta')^3 \int_{\mathbb{R}^3} \varphi(\theta'(y - z))Y(t, z)dz,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^3)$ satisfies

$$\widehat{\varphi}(\xi) = 1 \text{ for } |\xi| \leq \frac{1}{2}, \quad \widehat{\varphi}(\xi) = 0 \text{ for } |\xi| \geq 1,$$

so that

$$\int_{\mathbb{R}^3} \varphi(y)dy = 1, \quad \int_{\mathbb{R}^3} y^\alpha \varphi(y)dy = 0, \quad \forall |\alpha| > 0.$$

We then have

$$|S^{(2)}(\theta')Y|_{k,N} \leq C_{N,M}(\theta')^{N-M}|Y|_{k,M} \quad \text{if } N \geq M \geq 0,$$

as well as

$$|(1 - S^{(2)}(\theta'))Y|_{k,M} \leq C_{N,M}(\theta')^{-(N-M)}|Y|_{k,N} \quad \text{if } N \geq M \geq 0.$$

Define the operator

$$(2.62) \quad S(\theta, \theta') \stackrel{\text{def}}{=} S^{(1)}(\theta)S^{(2)}(\theta'), \quad \text{for } \theta, \theta' \geq 1.$$

Then it follows that

$$(2.63) \quad \begin{aligned} |S(\theta, \theta')Y|_{k,N} &\leq C\theta^{k-s}(\theta')^{N-M}|Y|_{s,M}, \\ \|\langle t \rangle^k S(\theta, \theta')g\|_{L_t^p(H^N)} &\leq C\theta^{k-s}(\theta')^{N-M}\|\langle t \rangle^s g\|_{L_t^p(H^M)} \quad \text{if } k \geq s \geq 0, N \geq M \geq 0. \end{aligned}$$

Moreover, due to

$$1 - S(\theta, \theta') = (1 - S^{(1)}(\theta)) + S^{(1)}(\theta)(1 - S^{(2)}(\theta')),$$

one has

$$(2.64) \quad \begin{aligned} |(1 - S(\theta, \theta'))Y|_{s,M} &\leq C\theta^{-(k-s)}|Y|_{k,M} + C(\theta')^{-(N-M)}|Y|_{s,N}, \\ \|\langle t \rangle^s (1 - S(\theta, \theta'))g\|_{L_t^p(H^M)} &\leq C\theta^{-(k-s)}\|\langle t \rangle^k g\|_{L_t^p(H^M)} + C(\theta')^{-(N-M)}\|\langle t \rangle^s g\|_{L_t^p(H^N)} \end{aligned}$$

provided that $k \geq s \geq 0, N \geq M \geq 0$.

Let us denote

$$\Phi(Y) \stackrel{\text{def}}{=} Y_{tt} - \Delta Y_t - \partial_3^2 Y - f(Y),$$

for f given by (2.25). Then we can write (2.1) equivalently as

$$(2.65) \quad \Phi(Y) = 0, \quad Y(0, y) = Y^{(0)}, \quad Y_t(0, y) = Y^{(1)}.$$

We aim to solve (2.65) via Nash-Moser iteration scheme in Section 10.

Let us define Y_0 via

$$(2.66) \quad \begin{cases} \partial_{tt}Y_0 - \Delta\partial_tY_0 - \partial_3^2Y_0 = 0, \\ Y_0(0, y) = Y^{(0)}, \quad \partial_tY_0(0, y) = Y^{(1)}. \end{cases}$$

Inductively, assume that we already determine Y_p . In order to define Y_{p+1} , we introduce a mollified version of $\Phi'(Y_p)$ as follows

$$(2.67) \quad L_p X \stackrel{\text{def}}{=} \Phi'(S_p Y_p) X = X_{tt} - \Delta X_t - \partial_3^2 X - f'(S_p Y_p; X),$$

where S_p is the smoothing operator defined by

$$(2.68) \quad S_p = S(\theta_p, \theta'_p), \quad \text{with} \quad \theta_p = 2^p, \quad \theta'_p = \theta_p^{\bar{\varepsilon}} = 2^{\bar{\varepsilon}p}, \quad \text{and} \quad p \geq 0,$$

where $S(\theta, \theta')$ is defined in (2.62) and $\bar{\varepsilon} > 0$ is a small constant to be chosen later on. Then it follows from (2.63) and (2.64) that

$$(S \text{ I}) \quad \begin{aligned} |S_p Y|_{k,N} &\leq C \theta_p^{k-s} \theta_p^{\bar{\varepsilon}(N-M)} |Y|_{s,M} \\ \|\langle t \rangle^k S_p g\|_{L_t^2(H^N)} &\leq C \theta_p^{k-s} \theta_p^{\bar{\varepsilon}(N-M)} \|\langle t \rangle^s g\|_{L_t^2(H^M)} \\ \|S_p g\|_{L_t^1(\delta, N)} &\leq C \theta_p^{\bar{\varepsilon}(N-M)} \|g\|_{L_t^1(\delta, M)}, \end{aligned}$$

and

$$(S \text{ II}) \quad \begin{aligned} |(1 - S_p)Y|_{0,0} &\leq C_{k,N} (\theta_p^{-k} |Y|_{k,0} + \theta_p^{-\bar{\varepsilon}N} |Y|_{0,N}), \\ \|\langle t \rangle^s (1 - S_p)g\|_{L_t^2(L^2)} &\leq C_{k,N} (\theta_p^{-(k-s)} \|\langle t \rangle^k g\|_{L_t^2(L^2)} + \theta_p^{-\bar{\varepsilon}N} \|\langle t \rangle^s g\|_{L_t^2(H^N)}), \end{aligned}$$

for $k \geq s \geq 0$, $N \geq M \geq 0$, where the norm $\|\cdot\|_{L_t^1(\delta, N)}$ is given by (2.39).

Remark 2.1. According to Remark 4.1 below, we can write

$$f'(S_p Y_p; X) = f'_0(S_p Y_p; X) + f'_1(S_p Y_p; X) + f'_2(S_p Y_p; X)$$

where

$$\begin{aligned} f'_0(S_p Y_p; X) &= F'_{0,U}(S_p \nabla \partial_t Y_p, S_p \nabla Y_p) \nabla X_t + F'_{0,V}(S_p \nabla \partial_t Y_p, S_p \nabla Y_p) \nabla X, \\ f'_1(S_p Y_p; X) &= F'_U(S_p \partial_3 Y_p, S_p \nabla Y_p) \partial_3 X + F'_V(S_p \partial_3 Y_p, S_p \nabla Y_p) \nabla X, \\ f'_2(S_p Y_p; X) &= F'_U(S_p \partial_t Y_p, S_p \nabla Y_p) X_t + F'_V(S_p \partial_t Y_p, S_p \nabla Y_p) \nabla X, \end{aligned}$$

where the functionals F'_0, F' will be presented in Remark 4.1.

Following Hörmander's version of Nash-Moser Scheme ([17]) (see also Klainerman's seminar papers [18, 19]), we define

$$(2.69) \quad Y_{p+1} = Y_p + X_p, \quad \text{with} \quad X_p = L_p^{-1} g_p,$$

where L_p^{-1} is a right inverse operator of L_p with zero initial data, that is: $X = L_p^{-1} g_p$ solves

$$(2.70) \quad \begin{cases} L_p X = g_p & \text{with } L_p \text{ given by (2.67),} \\ X(0, y) = 0, \quad X_t(0, y) = 0. \end{cases}$$

In order to prove the convergence of the scheme, we define

$$(2.71) \quad e'_p \stackrel{\text{def}}{=} (\Phi'(Y_p) - L_p) X_p, \quad e''_p \stackrel{\text{def}}{=} \Phi(Y_{p+1}) - \Phi(Y_p) - \Phi'(Y_p) X_p, \quad \text{and} \quad e_p \stackrel{\text{def}}{=} e'_p + e''_p,$$

from which, we infer

$$\begin{aligned}\Phi(Y_{p+1}) - \Phi(Y_p) &= \Phi'(Y_p)X_p + e_p'' = \Phi'(Y_p)L_p^{-1}g_p + e_p'' \\ &= (\Phi'(Y_p) - L_p)L_p^{-1}g_p + g_p + e_p'' = e_p' + e_p'' + g_p.\end{aligned}$$

As a result, it comes out

$$(2.72) \quad \Phi(Y_{p+1}) - \Phi(Y_p) = e_p + g_p \quad \text{and} \quad \Phi(Y_{p+1}) - \Phi(Y_0) = \sum_{j=0}^p (e_j + g_j).$$

To achieve that the above limit as $p \rightarrow \infty$ is equal to $-\Phi(Y_0)$, we set

$$(2.73) \quad \sum_{j=0}^p g_j + S_p E_p = -S_p \Phi(Y_0) \quad \text{with} \quad E_p \stackrel{\text{def}}{=} \sum_{j=0}^{p-1} e_j.$$

The last relation defines g_p as follows

$$(2.74) \quad g_0 = -S_0 \Phi(Y_0), \quad \text{and} \quad g_p = -(S_p - S_{p-1})E_{p-1} - S_p e_{p-1} - (S_p - S_{p-1})\Phi(Y_0).$$

Remark 2.2. By virtue of Remarks 2.1, 4.1, 4.2, using a Taylor formula to (2.71), we have

$$\begin{aligned}e_p' &= - \int_0^1 f''(sY_p + (1-s)S_p Y_p; (1-S_p)Y_p, X_p) ds, \quad \text{and} \\ e_p'' &= - \int_0^1 (1-s)f''(sY_{p+1} + (1-s)Y_p; X_p, X_p) ds,\end{aligned}$$

where f'' should be understood in the way explained in Remark 4.2. Then we have

$$(2.75) \quad \begin{aligned}e_p &= e_{p,0} + e_{p,1} + e_{p,2}, \quad \text{with} \quad e_{p,m} \stackrel{\text{def}}{=} e_{p,m}' + e_{p,m}'' \quad \text{and} \\ e_{p,m}' &\stackrel{\text{def}}{=} - \int_0^1 f_m''(sY_p + (1-s)S_p Y_p; (1-S_p)Y_p, X_p) ds, \\ e_{p,m}'' &\stackrel{\text{def}}{=} - \int_0^1 (1-s)f_m''(Y_p + sX_p; X_p, X_p) ds, \quad m = 0, 1, 2.\end{aligned}$$

Let us fix the small constants: $\varepsilon, \bar{\varepsilon}, \delta > 0$, so that

$$(2.76) \quad \bar{\varepsilon} \leq \frac{1}{20}, \quad \delta + 5\bar{\varepsilon} \leq \frac{1}{4}, \quad \delta + \varepsilon + 4\bar{\varepsilon} \leq \frac{1}{4}.$$

Let us take

$$(2.77) \quad \gamma = \frac{1}{4} - \bar{\varepsilon}, \quad \beta = \frac{1}{4} + \bar{\varepsilon},$$

and $N_0 \in \mathbb{N}$ is chosen such that

$$(2.78) \quad \bar{\varepsilon}N_0 \geq \frac{1}{2} = \gamma + \beta.$$

In Section 10, we shall inductively prove the following statements:

Proposition 2.6. Let $\delta_1 > 0$ be determined by Propositions 2.3, 2.4, 8.1, 8.2, 9.1, 9.2 and Theorem 2.3. Then for the constants $\beta, \gamma, N_0, \varepsilon, \bar{\varepsilon}$ and δ given by (2.76-2.78), for any $0 \leq N \leq N_0$, we have

$$(P1, p) \quad \begin{aligned} & \| |D|^{-1}(\partial_3 X_p, \partial_t X_p) \|_{0, N+2} + \| \nabla X_p \|_{0, N+1} + \| (\partial_t X_p, \partial_3 X_p) \|_{\frac{1}{2}, N+1} \\ & + \| \partial_t X_p \|_{L_t^2(H^{N+2})} + \| (\partial_3 X_p, \langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p) \|_{L_t^2(H^{N+1})} + \| \nabla \partial_t X_p \|_{1, N-1} \leq \eta \theta_p^{-\beta + \bar{\varepsilon}N}; \end{aligned}$$

and

$$\begin{aligned}
 (P2, p) \quad & |\partial_3 X_p|_{k,N} \leq \eta \theta_p^{k-\frac{1}{2}-\gamma+\bar{\varepsilon}N} \quad \text{if } \frac{1}{2} \leq k \leq 1, \\
 & |\partial_t X_p|_{k,N} \leq \eta \theta_p^{k-(1-\delta)-\gamma+\bar{\varepsilon}N} \quad \text{if } 1-\delta \leq k \leq \frac{3}{2}-\delta, \\
 & |X_p|_{k,N} \leq \eta \theta_p^{k-\gamma+\bar{\varepsilon}N} \quad \text{if } 0 \leq k \leq \frac{1}{2};
 \end{aligned}$$

and

$$\begin{aligned}
 (P3, p) \quad & \|\nabla Y_p\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \leq \delta_1, \quad \|\nabla Y_p\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{5}{2}})} \leq 1, \quad \|\partial_t Y_p\|_{0,0} \leq 1, \quad |\partial_t Y_p|_{0,1} \leq 1, \\
 & |\partial_3 Y_p|_{\frac{1}{2}+\varepsilon,1}^{\frac{4}{3}} \|\partial_3 Y_p\|_{L_t^2(L^2)}^{\frac{2}{3}} + |\partial_3 Y_p|_{\frac{1}{2}+\varepsilon,1}^2 + |\partial_t Y_p|_{1+\varepsilon,2} \leq 1.
 \end{aligned}$$

Recall the convention that $\|u\|_{k,-1} = 0$. We shall deduce the following propositions from Proposition 2.6.

Proposition 2.7. *Under the assumptions of Proposition 2.6, we have, for $N \geq 0$,*

$$\begin{aligned}
 (I) \quad & |S_{p+1} \partial_3 Y_{p+1}|_{k,N} \leq C_{k,N} \eta \theta_{p+1}^{k-\frac{1}{2}-\gamma+\bar{\varepsilon}N} \quad \text{if } k \geq \frac{1}{2}, \quad k - \frac{1}{2} - \gamma + \bar{\varepsilon}N \geq \bar{\varepsilon}, \\
 (i) \quad & |S_{p+1} \partial_t Y_{p+1}|_{k,N} \leq C_{k,N} \eta \theta_{p+1}^{k-(1-\delta)-\gamma+\bar{\varepsilon}N} \quad \text{if } k \geq 1-\delta, \quad k - (1-\delta) - \gamma + \bar{\varepsilon}N \geq \bar{\varepsilon}, \\
 & |S_{p+1} Y_{p+1}|_{k,N} \leq C_{k,N} \eta \theta_{p+1}^{k-\gamma+\bar{\varepsilon}N} \quad \text{if } k \geq 0, \quad k - \gamma + \bar{\varepsilon}N \geq \bar{\varepsilon}; \\
 (ii) \quad & \Delta_{p+1} \stackrel{\text{def}}{=} \| |D|^{-1} S_{p+1} (\partial_3 Y_{p+1}, \partial_t Y_{p+1}) \|_{0,N+2} + \| S_{p+1} \nabla Y_{p+1} \|_{0,N+1} \\
 & + \| S_{p+1} (\partial_t Y_{p+1}, \partial_3 Y_{p+1}) \|_{\frac{1}{2},N+1} + \| (S_{p+1} \partial_3 Y_{p+1}, \langle t \rangle^{\frac{1}{2}} S_{p+1} \nabla \partial_t Y_{p+1}) \|_{L_t^2(H^{N+1})} \\
 & + \| S_{p+1} \partial_t Y_{p+1} \|_{L_t^2(H^{N+2})} + \| S_{p+1} \nabla \partial_t Y_{p+1} \|_{1,N-1} \leq C_N \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}N} \quad \text{if } -\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}; \\
 (II) \quad & |S_{p+1} \partial_3 Y_{p+1}|_{k,N} \leq C_{k,N} \eta \quad \text{if } k \geq \frac{1}{2}, \quad k - \frac{1}{2} - \gamma + \bar{\varepsilon}N \leq -\bar{\varepsilon}, \\
 (i) \quad & |S_{p+1} \partial_t Y_{p+1}|_{k,N} \leq C_{k,N} \eta \quad \text{if } k \geq 1-\delta, \quad k - (1-\delta) - \gamma + \bar{\varepsilon}N \leq -\bar{\varepsilon}, \\
 & |S_{p+1} Y_{p+1}|_{k,N} \leq C_{k,N} \eta \quad \text{if } k \geq 0, \quad k - \gamma + \bar{\varepsilon}N \leq -\bar{\varepsilon}; \\
 (ii) \quad & \Delta_{p+1} \leq C_N \eta \quad \text{if } -\beta + \bar{\varepsilon}N \leq -\bar{\varepsilon};
 \end{aligned}$$

$$\begin{aligned}
 (III) \quad & |(1 - S_{p+1}) \partial_3 Y_{p+1}|_{k,N} \leq C_{k,N} \eta \theta_{p+1}^{k-\frac{1}{2}-\gamma+\bar{\varepsilon}N} \quad \text{if } \frac{1}{2} \leq k \leq 1, \quad N \leq N_0, \\
 & |(1 - S_{p+1}) \partial_t Y_{p+1}|_{k,N} \leq C_{k,N} \eta \theta_{p+1}^{k-(1-\delta)-\gamma+\bar{\varepsilon}N} \quad \text{if } 1-\delta \leq k \leq \frac{3}{2}-\delta, \quad N \leq N_0, \\
 & |(1 - S_{p+1}) Y_{p+1}|_{k,N} \leq C_{k,N} \eta \theta_{p+1}^{k-\gamma+\bar{\varepsilon}N} \quad \text{if } 0 \leq k \leq \frac{1}{2}, \quad N \leq N_0.
 \end{aligned}$$

Proposition 2.8. *Let e_p , g_p and $R_{N,\theta}(g)$ be given by (2.71), (2.74) and (2.38) respectively. Let $\alpha \stackrel{\text{def}}{=} \frac{1}{2} - \delta - \bar{\varepsilon} > 0$. Then there hold*

(1) Estimates for e_p .

$$\begin{aligned}
 (IV) \quad & (i) \quad \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} e_p\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} \quad \text{if } 0 \leq k \leq \alpha, \quad 0 \leq N \leq N_0 - 2, \\
 & (ii) \quad \| |D|^{-1} e_p \|_{1+k,N+1} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)} \quad \text{if } 0 \leq k \leq \frac{1}{2} - \delta, \quad N \leq N_0 - 2, \\
 & (iii) \quad \|\langle t \rangle^{\frac{1}{2}} e_p\|_{L_t^2(\delta,N)} \lesssim \eta^2 \theta_p^{-\gamma+\bar{\varepsilon}(N+5)} \quad \text{if } 0 \leq N \leq N_0 - 6;
 \end{aligned}$$

(2) Estimates for g_{p+1} .

- (V) (i) $\|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} g_{p+1}\|_{L_t^2(H^{N+1})} \leq C \eta^2 \theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}$ if $k \geq 0, N \geq 0$,
(V) (ii) $\||D|^{-1} g_{p+1}\|_{1+k, N+1} \lesssim \eta^2 \theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}$ if $k \geq 0, N \geq 0$,
(V) (iii) $\|g_{p+1}\|_{L_t^1(N)} \leq C \eta^2 \theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+6)}$ if $-\gamma + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}$,
(V) (iv) $\|g_{p+1}\|_{L_t^1(N)} \leq C \eta^2 \theta_{p+1}^{\bar{\varepsilon}}$ if $-\gamma + \bar{\varepsilon}(N+5) \leq -\bar{\varepsilon}$;

(3) Estimates for $R_{N, \theta_{p+1}}(g_{p+1})$.

- (VI) (i) $R_{N, \theta_{p+1}}(g_{p+1}) \leq C \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N}$ if $-\gamma + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}$,
(VI) (ii) $R_{0, \theta_{p+1}}(g_{p+1}) \leq C \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma}$.

The following interpolation lemma will be crucial in the proof of the above propositions, whose proof is exactly the same as that of Lemma 6.1 of [18], which we omit the details here.

Lemma 2.1 (Interpolation lemma). *Let $p \in [1, +\infty]$, $\theta \geq 1$ and $\bar{\varepsilon} > 0$, which satisfy*

$$\beta > \bar{\varepsilon}, \quad k_0 - \beta \geq \bar{\varepsilon}, \quad -\beta + \bar{\varepsilon}N_0 \geq \bar{\varepsilon}.$$

Assume that $u \in C^\infty([0, +\infty) \times \mathbb{R}^n)$ satisfies

$$(2.79) \quad \begin{aligned} \|u\|_{L_t^p(L^2)} &\leq C \theta^{-\beta}, \\ \|\langle t \rangle^k u\|_{L_t^p(H^N)} &\leq C \theta^{k-\beta+\bar{\varepsilon}N}, \text{ for } 0 \leq k \leq k_0, 0 \leq N \leq N_0 \text{ s.t. } k - \beta + \bar{\varepsilon}N \geq \bar{\varepsilon}. \end{aligned}$$

Then for all $0 \leq k \leq k_0, 0 \leq N \leq N_0$,

$$\|\langle t \rangle^k u\|_{L_t^p(H^N)} \leq C_{k_0, N_0} \theta^{k-\beta+\bar{\varepsilon}N}.$$

Finally with the previous propositions, we shall prove the convergence of the approximate solutions constructed by (2.69) in Subsection 10.4, and this completes the proof of Theorem 2.1.

3. DECAY ESTIMATES OF THE LINEAR EQUATION

3.1. Decay estimates for the solution operator. Following the strategy in [18, 19], we first investigate the decay properties of the solutions to the linear equation (2.31) with $\mathcal{Y}_0 = 0$ and $\mathcal{Y}_1 = Y_1$. By taking Fourier transform to (2.31) with respect to y variables and solving the resulting ODE, we write

$$(3.1) \quad \mathcal{Y}(t, y) = \Gamma(t, D) Y_1 \quad \text{with} \quad \Gamma(t, \xi) = \frac{1}{\lambda_2(\xi) - \lambda_1(\xi)} \left(e^{t\lambda_2(\xi)} - e^{t\lambda_1(\xi)} \right)$$

where $\lambda_1(\xi)$ and $\lambda_2(\xi)$ are given by (2.2).

Proposition 3.1. *Given $\delta \in [0, 1[$ and $N \in \mathbb{N}$, there exists $C_{\delta, N} > 0$ such that there holds*

$$(3.2) \quad |\partial_3 \Gamma(t) Y_1|_{1, N} + |\partial_3^2 \Gamma(t) Y_1|_{\frac{3}{2}, N} + |\partial_t \Gamma(t) Y_1|_{\frac{3}{2}-\delta, N} + |\Gamma(t) Y_1|_{\frac{1}{2}, N} \leq C_{\delta, N} \|(|D|^{2\delta} Y_1, |D|^{N+4} Y_1)\|_{L^1}.$$

Proof. The estimate (3.2) for general $N \in \mathbb{N}$ follows from the case when $N = 0$. Due to the anisotropic properties of the eigenvalues $\lambda_1(\xi), \lambda_2(\xi)$, we shall split the frequency space into two parts: $\{\xi \in \mathbb{R}^3 : |\xi|^2 \geq 2|\xi_3|\}$ and $\{\xi \in \mathbb{R}^3 : |\xi|^2 < 2|\xi_3|\}$. When $|\xi|^2 \geq 2|\xi_3|$, let us denote $\alpha(\xi) \stackrel{\text{def}}{=} \sqrt{\frac{|\xi|^4}{4} - \xi_3^2}$. Then we have

$$\lambda_1(\xi) = -\frac{|\xi|^2}{2} + \alpha(\xi) \quad \text{and} \quad \lambda_2(\xi) = -\frac{|\xi|^2}{2} - \alpha(\xi),$$

and we write

$$(3.3) \quad \Gamma(t, \xi) \mathbf{1}_{|\xi|^2 \geq 2|\xi_3|} = e^{-t\left(\frac{|\xi|^2}{2} - \alpha(\xi)\right)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)} \mathbf{1}_{|\xi|^2 \geq 2|\xi_3|}.$$

When $|\xi|^2 < 2|\xi_3|$, let us denote $\beta(\xi) \stackrel{\text{def}}{=} \sqrt{\xi_3^2 - \frac{|\xi|^4}{4}}$. Then we have

$$\lambda_1(\xi) = -\frac{|\xi|^2}{2} + i\beta(\xi) \quad \text{and} \quad \lambda_2(\xi) = -\frac{|\xi|^2}{2} - i\beta(\xi),$$

and we write

$$(3.4) \quad \Gamma(t, \xi) \mathbf{1}_{|\xi|^2 < 2|\xi_3|} = e^{-\frac{t}{2}|\xi|^2} \frac{\sin(t\beta(\xi))}{\beta(\xi)} \mathbf{1}_{|\xi|^2 < 2|\xi_3|}.$$

Next we handle the estimate of (3.2) term by term below.

• *Estimates of $\|\partial_3 \mathcal{Y}(t)\|_{L^\infty}$ and $\|\partial_3^2 \mathcal{Y}(t)\|_{L^\infty}$.*

In view of (3.1), we deduce that

$$(3.5) \quad \begin{aligned} \|\partial_3 \mathcal{Y}(t)\|_{L^\infty} &\leq \|\Gamma(t, \cdot) \xi_3 \widehat{Y_1}(\cdot)\|_{L^1} \\ &= \int_{|\xi|^2 \geq 2|\xi_3|} e^{-t\left(\frac{|\xi|^2}{2} - \alpha(\xi)\right)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\xi_3 \widehat{Y_1}(\xi)| d\xi \\ &\quad + \int_{|\xi|^2 < 2|\xi_3|} e^{-\frac{t}{2}|\xi|^2} \frac{|\sin(t\beta(\xi))|}{\beta(\xi)} |\xi_3 \widehat{Y_1}(\xi)| d\xi \stackrel{\text{def}}{=} I_1 + I_2. \end{aligned}$$

It is easy to observe that

$$I_1 = \left(\int_{|\xi| \geq 3} + \int_{9 > |\xi|^2 \geq 2|\xi_3|} \right) e^{-t\left(\frac{|\xi|^2}{2} - \alpha(\xi)\right)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\xi_3 \widehat{Y_1}(\xi)| d\xi,$$

and

$$\begin{aligned} &\int_{|\xi| \geq 3} e^{-t\left(\frac{|\xi|^2}{2} - \alpha(\xi)\right)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\xi_3 \widehat{Y_1}(\xi)| d\xi \\ &\leq \| |\xi|^3 \widehat{Y_1} \|_{L^\infty} \int_{|\xi| \geq 3} e^{-\frac{t}{2} \frac{\xi_3^2}{|\xi|^2} + \alpha(\xi)} \frac{|\xi_3|}{2\alpha(\xi) |\xi|^3} d\xi \\ &\leq 2 \| |\xi|^3 \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_3^\infty e^{-t \cos^2 \phi} \frac{1}{r \sqrt{r^2 - 4 \cos^2 \phi}} \sin \phi \cos \phi d\phi dr \\ &\leq C \| |\xi|^3 \widehat{Y_1} \|_{L^\infty} \int_0^1 e^{-t\tau} \int_3^\infty \frac{1}{r \sqrt{r^2 - 4\tau^2}} dr d\tau \\ &\leq C \langle t \rangle^{-1} \| |D|^3 Y_1 \|_{L^1}. \end{aligned}$$

Exactly along the same line, we have

$$\begin{aligned} &\int_{9 > |\xi|^2 \geq 2|\xi_3|} e^{-t\left(\frac{|\xi|^2}{2} - \alpha(\xi)\right)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\xi_3 \widehat{Y_1}(\xi)| d\xi \\ &\leq 2 \| |\xi| \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_{2 \cos \phi}^3 e^{-t \cos^2 \phi} \frac{\sin \phi \cos \phi}{\sqrt{r^2 - 4 \cos^2 \phi}} r dr d\phi \\ &\leq C \| |\xi| \widehat{Y_1} \|_{L^\infty} \int_0^1 e^{-t\tau} \int_{2\sqrt{\tau}}^3 \frac{r}{\sqrt{r^2 - 4\tau}} dr d\tau \\ &\leq C \langle t \rangle^{-1} \| |D| Y_1 \|_{L^1}. \end{aligned}$$

This proves

$$(3.6) \quad I_1 \leq C\langle t \rangle^{-1} (\| |D| Y_1 \|_{L^1} + \| |D|^3 Y_1 \|_{L^1}).$$

The estimate of I_2 is much simpler. By virtue of (3.5), we have

$$(3.7) \quad \begin{aligned} I_2 &\leq 2 \| |\xi| \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} e^{-\frac{t}{2}r^2} \frac{1}{\sqrt{4\cos^2\phi - r^2}} \sin\phi \cos\phi r dr d\phi \\ &\leq 2 \| |\xi| \widehat{Y_1} \|_{L^\infty} \int_0^1 e^{-\frac{t}{2}r^2} r \int_{\frac{r^2}{4}}^1 \frac{1}{\sqrt{4\tau - r^2}} d\tau dr \\ &\leq C\langle t \rangle^{-1} \| |D| Y_1 \|_{L^1}. \end{aligned}$$

As a result, we achieve

$$(3.8) \quad \|\partial_3 \mathcal{Y}(t)\|_{L^\infty} \leq C\langle t \rangle^{-1} (\| |D| Y_1 \|_{L^1} + \| |D|^3 Y_1 \|_{L^1}).$$

Along the same line to the proof of (3.8), we infer

$$\begin{aligned} \|\partial_3^2 \mathcal{Y}(t)\|_{L^\infty} &\leq 2 \| |\xi|^4 \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_3^\infty e^{-t\cos^2\phi} \frac{1}{r\sqrt{r^2 - 4\cos^2\phi}} \sin\phi \cos^2\phi d\phi dr \\ &\quad + 2 \| |\xi|^2 \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_{2\cos\phi}^3 e^{-t\cos^2\phi} \frac{\sin\phi \cos^2\phi}{\sqrt{r^2 - 4\cos^2\phi}} r dr d\phi \\ &\quad + 2 \| |\xi| \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} e^{-\frac{t}{2}r^2} \frac{1}{\sqrt{4\cos^2\phi - r^2}} \sin\phi \cos^2\phi r^2 dr d\phi, \end{aligned}$$

so that for t large enough, there holds

$$\begin{aligned} \|\partial_3^2 \mathcal{Y}(t)\|_{L^\infty} &\leq C t^{-\frac{1}{2}} \left(\| |\xi|^4 \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_3^\infty e^{-\frac{t\cos^2\phi}{2}} \frac{1}{r\sqrt{r^2 - 4\cos^2\phi}} \sin\phi \cos\phi d\phi dr \right. \\ &\quad \left. + \| |\xi|^2 \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \int_{2\cos\phi}^3 e^{-\frac{t\cos^2\phi}{2}} \frac{\sin\phi \cos\phi}{\sqrt{r^2 - 4\cos^2\phi}} r dr d\phi \right) \\ &\quad + \| |\xi| \widehat{Y_1} \|_{L^\infty} \int_0^1 e^{-\frac{t}{2}r^2} r^2 \int_{\frac{r^2}{4}}^1 \frac{1}{\sqrt{4\tau - r^2}} d\tau dr. \end{aligned}$$

This gives rise to

$$(3.9) \quad \|\partial_3^2 \mathcal{Y}(t)\|_{L^\infty} \leq C\langle t \rangle^{-\frac{3}{2}} (\| |D| Y_1 \|_{L^1} + \| |D|^4 Y_1 \|_{L^1}).$$

• Estimate of $|\partial_t \mathcal{Y}(t)|_{L^\infty}$.

It follows from (3.1) that

$$\partial_t \Gamma(t, \xi) = \frac{1}{\lambda_2(\xi) - \lambda_1(\xi)} \left(\lambda_2(\xi) e^{t\lambda_2(\xi)} - \lambda_1(\xi) e^{t\lambda_1(\xi)} \right).$$

So that one has

$$(3.10) \quad \begin{aligned} \partial_t \Gamma(t, \xi) \mathbf{1}_{|\xi|^2 \geq 2|\xi_3|} &= e^{-t\left(\frac{|\xi|^2}{2} + \alpha(\xi)\right)} - e^{-t\left(\frac{|\xi|^2}{2} - \alpha(\xi)\right)} \left(\frac{|\xi|^2}{2} - \alpha(\xi) \right) \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)}, \\ \partial_t \Gamma(t, \xi) \mathbf{1}_{|\xi|^2 < 2|\xi_3|} &= e^{-\frac{t}{2}|\xi|^2} \left(-\frac{|\xi|^2}{2} \frac{\sin(t\beta(\xi))}{\beta(\xi)} + \cos(t\beta(\xi)) \right). \end{aligned}$$

It is easy to observe that for any $\delta \in [0, 1[$,

$$\begin{aligned} \int_{\mathbb{R}^3} e^{-\frac{t}{2}|\xi|^2} |\widehat{Y_1}(\xi)| d\xi &\leq \| |\xi|^{2\delta} \widehat{Y_1} \|_{L^\infty} \int_{\mathbb{R}^3} |\xi|^{-2\delta} e^{-\frac{t}{2}|\xi|^2} d\xi \\ &\leq C t^{-(\frac{3}{2}-\delta)} \| |D|^{2\delta} Y_1 \|_{L^1}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} e^{-\frac{t}{2}|\xi|^2} |\widehat{Y_1}(\xi)| d\xi &\leq \| |\xi|^{2\delta} \widehat{Y_1} \|_{L^\infty} \int_{|\xi| \leq 1} |\xi|^{-2\delta} d\xi + \| |\xi|^4 \widehat{Y_1} \|_{L^\infty} \int_{|\xi| > 1} |\xi|^{-4} d\xi \\ &\leq C(\| |D|^{2\delta} Y_1 \|_{L^1} + \| |D|^4 Y_1 \|_{L^1}). \end{aligned}$$

This leads to

$$\int_{\mathbb{R}^3} e^{-\frac{t}{2}|\xi|^2} |\widehat{Y_1}(\xi)| d\xi \leq C \langle t \rangle^{-(\frac{3}{2}-\delta)} (\| |D|^{2\delta} Y_1 \|_{L^1} + \| |D|^4 Y_1 \|_{L^1}).$$

While similar to estimate of (3.6) and (3.7), we infer

$$\begin{aligned} &\int_{|\xi|^2 \geq 2|\xi_3|} e^{-\frac{t}{2} \frac{|\xi_3|^2}{|\xi|^2 + \alpha(\xi)}} \frac{\xi_3^2}{\frac{|\xi|^2}{2} + \alpha(\xi)} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\widehat{Y_1}(\xi)| d\xi \\ &\leq 2 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{2\cos\phi}^\infty e^{-t\cos^2\phi} 2\cos^2\phi \frac{|\widehat{Y_1}(\xi(r, \theta, \phi))|}{\sqrt{r^2 - 4\cos^2\phi}} \sin\phi r dr d\theta d\phi \\ &\leq 2 \| |\xi|^{2\delta} \widehat{Y_1} \|_{L^\infty} \int_0^1 e^{-t\tau} \sqrt{\tau} \int_{2\sqrt{\tau}}^3 \frac{r^{1-2\delta}}{\sqrt{r^2 - 4\tau}} dr d\tau \\ &\quad + 2 \| |\xi|^2 \widehat{Y_1} \|_{L^\infty} \int_0^1 e^{-t\tau} \sqrt{\tau} \int_3^\infty \frac{1}{r\sqrt{r^2 - 4\tau}} dr d\tau \\ &\leq C \langle t \rangle^{-\frac{3}{2}} (\| |D|^{2\delta} Y_1 \|_{L^1} + \| |D|^2 Y_1 \|_{L^1}), \end{aligned}$$

and

$$\begin{aligned} \int_{|\xi|^2 \leq 2|\xi_3|} e^{-\frac{t}{4}|\xi|^2} \frac{|\xi_3|}{\beta(\xi)} |\widehat{Y_1}(\xi)| d\xi &\leq 2 \| |\xi|^{2\delta} \widehat{Y_1} \|_{L^\infty} \int_0^{\frac{\pi}{2}} \frac{2\sin\phi \cos\phi}{\sqrt{4\cos^2\phi - r^2}} \int_0^{2\cos\phi} e^{-\frac{t}{4}r^2} r^{2(1-\delta)} dr d\phi \\ &\leq C \langle t \rangle^{-(\frac{3}{2}-\delta)} \| |\xi|^{2\delta} \widehat{Y_1} \|_{L^\infty} \leq C \langle t \rangle^{-(\frac{3}{2}-\delta)} \| |D|^{2\delta} Y_1 \|_{L^1}. \end{aligned}$$

Hence by virtue of (3.10), we obtain

$$(3.11) \quad \|\partial_t \mathcal{Y}(t)\|_{L^\infty} \leq C \langle t \rangle^{-(\frac{3}{2}-\delta)} (\| |D|^{2\delta} Y_1 \|_{L^1} + \| |D|^4 Y_1 \|_{L^1}).$$

• Estimate of $\|\mathcal{Y}(t)\|_{L^\infty}$.

Note that

$$\int_0^1 \int_{2\tau}^3 e^{-t\tau^2} \frac{r^{\frac{1}{2}}}{\sqrt{r^2 - 4\tau^2}} dr d\tau \leq \int_0^1 e^{-t\tau^2} \int_{2\tau}^3 (r - 2\tau)^{-\frac{1}{2}} dr d\tau \leq C \langle t \rangle^{-\frac{1}{2}}.$$

we find

$$\begin{aligned} &\int_{|\xi|^2 \geq 2|\xi_3|} e^{-t(\frac{|\xi|^2}{2} - \alpha(\xi))} \frac{1 - e^{-2t\alpha(\xi)}}{2\alpha(\xi)} |\widehat{Y_1}(\xi)| d\xi \\ &\leq \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{2\cos\phi}^\infty e^{-t\cos^2\phi} \frac{|\widehat{Y_1}(\xi(r, \theta, \phi))|}{\frac{1}{2}r\sqrt{r^2 - 4\cos^2\phi}} r^2 \sin\phi dr d\theta d\phi \\ &= C \| |\xi|^{\frac{1}{2}} \widehat{Y_1} \|_{L^\infty} \int_0^1 \int_{2\tau}^3 e^{-t\tau^2} \frac{r^{\frac{1}{2}}}{\sqrt{r^2 - 4\tau^2}} dr d\tau + C \| |\xi|^2 \widehat{Y_1} \|_{L^\infty} \int_0^1 e^{-t\tau^2} \int_3^\infty \frac{1}{r\sqrt{r^2 - 4\tau^2}} dr d\tau \\ &\leq C \langle t \rangle^{-\frac{1}{2}} (\| |D|^{\frac{1}{2}} Y_1 \|_{L^1} + \| |D|^2 Y_1 \|_{L^1}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \int_{|\xi|^2 < 2|\xi_3|} e^{-t\frac{|\xi|^2}{2}} \frac{\sin(t\beta(\xi))}{2\beta(\xi)} |\widehat{Y_1}(\xi)| d\xi &\leq \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} e^{-\frac{t}{2}r^2} \frac{|\widehat{Y_1}(\xi(r, \theta, \phi))|}{r\sqrt{4\cos^2\phi - r^2}} r^2 \sin\phi dr d\phi \\ &\leq \| |\xi|^{\frac{1}{2}} \widehat{Y_1} \|_{L^\infty} \int_0^2 e^{-\frac{t}{2}r^2} \int_{r/2}^1 \frac{r^{\frac{1}{2}}}{\sqrt{4\tau^2 - r^2}} d\tau dr \\ &\leq C \langle t \rangle^{-\frac{1}{2}} \| |D|^{\frac{1}{2}} Y_1 \|_{L^1}. \end{aligned}$$

As a result, by virtue of (3.3) and (3.4), it comes out

$$(3.12) \quad \|\mathcal{Y}(t)\|_{L^\infty} \leq C \langle t \rangle^{-\frac{1}{2}} (\| |D|^{\frac{1}{2}} Y_1 \|_{L^1} + \| D^2 Y_1 \|_{L^1}).$$

(3.8) together with (3.9), (3.11) and (3.12) imply the estimate (3.2) for $N = 0$. \square

Lemma 3.1. For $N \in \mathbb{N}$, there exists $C_N > 0$ such that for $t > 0$,

$$(3.13) \quad \|t\Delta\partial_t\Gamma(t)Y_1\|_{L_t^\infty(H^N)} \leq C_N \|Y_1\|_N \quad \text{and} \quad \|t\nabla\partial_3^2\Gamma(t)Y_1\|_{L_t^\infty(H^N)} \leq C_N \|Y_1\|_{N+1}.$$

Proof. The two inequalities of (3.13) follows from the claim that

$$(3.14) \quad t|\xi|^2\partial_t\Gamma(t, \xi) \in L_t^\infty(L_\xi^\infty), \quad \text{and} \quad \frac{t|\xi|}{1+|\xi|} |\xi_3|^2\Gamma(t, \xi) \in L_t^\infty(L_\xi^\infty).$$

(1) When $|\xi|^2 \geq 2|\xi_3|$, we separate the proof of (3.14) into the following two cases:

- If $\frac{\sqrt{3}}{4}|\xi|^2 \leq |\xi_3| \leq \frac{1}{2}|\xi|^2$, we deduce from (3.10) that

$$\begin{aligned} |\partial_t\Gamma(t, \xi) \mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2 \leq |\xi_3| \leq \frac{1}{2}|\xi|^2}| &\leq e^{-t\frac{|\xi|^2}{4}} (1 + |\xi|^2 t), \\ |\xi_3^2\Gamma(t, \xi) \mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2 \leq |\xi_3| \leq \frac{1}{2}|\xi|^2}| &\leq t\xi_3^2 e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2} + \alpha(\xi)}} \mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2 \leq |\xi_3| \leq \frac{1}{2}|\xi|^2} \leq Ct\xi_3^2 e^{-ct|\xi_3|}. \end{aligned}$$

As a result, it comes out

$$t|\xi|^2|\partial_t\Gamma(t, \xi)| \mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2 \leq |\xi_3| \leq \frac{1}{2}|\xi|^2} \leq C \quad \text{and} \quad \frac{t|\xi|}{1+|\xi|} |\xi_3|^2\Gamma(t, \xi) \mathbf{1}_{\frac{\sqrt{3}}{4}|\xi|^2 \leq |\xi_3| \leq \frac{1}{2}|\xi|^2} \leq C.$$

- If $|\xi_3| \leq \frac{\sqrt{3}}{4}|\xi|^2$, then $\frac{|\xi|^2}{4} \leq \alpha(\xi) \leq \frac{|\xi|^2}{2}$, we deduce from (3.10) that

$$\begin{aligned} |\partial_t\Gamma(t, \xi) \mathbf{1}_{|\xi_3| \leq \frac{\sqrt{3}}{4}|\xi|^2}| &\leq e^{-t\frac{|\xi|^2}{2}} + e^{-t\frac{\xi_3^2}{|\xi|^2}} \frac{\xi_3^2}{|\xi|^4}, \\ \xi_3^2|\Gamma(t, \xi) \mathbf{1}_{|\xi_3| \leq \frac{\sqrt{3}}{4}|\xi|^2}| &\leq \frac{\xi_3^2}{\alpha(\xi)} e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2} + \alpha(\xi)}} \leq C \frac{\xi_3^2}{\frac{|\xi|^2}{2} + \alpha(\xi)} e^{-t\frac{\xi_3^2}{\frac{|\xi|^2}{2} + \alpha(\xi)}}, \end{aligned}$$

so that there holds

$$t|\xi|^2|\partial_t\Gamma(t, \xi)| \xi_3 \mathbf{1}_{|\xi_3| \leq \frac{\sqrt{3}}{4}|\xi|^2} \leq t|\xi|^2 e^{-t\frac{|\xi|^2}{2}} + t e^{-t\frac{\xi_3^2}{|\xi|^2}} \frac{\xi_3^2}{|\xi|^2} \leq C,$$

$$\frac{t|\xi|}{1+|\xi|} |\xi_3|^2\Gamma(t, \xi) \mathbf{1}_{|\xi_3| \leq \frac{\sqrt{3}}{4}|\xi|^2} \leq C.$$

(2) When $|\xi|^2 > 2|\xi_3|$, we infer from (3.10) that

$$|\partial_t\Gamma(t, \xi) \mathbf{1}_{|\xi|^2 > 2|\xi_3|}| \leq e^{-t\frac{|\xi|^2}{2}} (|\xi|^2 t + 1),$$

which implies

$$t|\xi|^2|\partial_t\Gamma(t, \xi) \mathbf{1}_{|\xi|^2 > 2|\xi_3|}| \leq C.$$

To prove the second estimate of (3.14), we divide further the region, $\{|\xi|^2 > 2|\xi_3|\}$, into two parts.

- If $|\xi|^2 \leq \sqrt{3}|\xi_3|$, then we have $\frac{|\xi_3|}{2} \leq \beta(\xi) \leq |\xi_3|$, and it follows from (3.4) that

$$\xi_3^2 |\Gamma(t, \xi)| \mathbf{1}_{|\xi|^2 \leq \sqrt{3}|\xi_3|} \leq C |\xi_3| e^{-t \frac{|\xi|^2}{2}} \leq \frac{C}{t |\xi|}.$$

- When $\sqrt{3}|\xi_3| < |\xi|^2 \leq 2|\xi_3|$, we have

$$\xi_3^2 |\Gamma(t, \xi)| \mathbf{1}_{\sqrt{3}|\xi_3| < |\xi|^2 \leq 2|\xi_3|} \leq C t |\xi_3|^2 e^{-ct|\xi_3|} \leq \frac{C}{t}.$$

By summarizing the above estimates, we obtain the second estimate of (3.14). This completes the proof of Lemma 3.1. \square

3.2. Energy estimates for the linear equation.

Lemma 3.2. *Let $\mathcal{Y}(t)$ be a smooth enough solution of the linear equation (2.31) with initial data $(\mathcal{Y}_0, \mathcal{Y}_1)$. Then for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that there hold (2.33) and (2.34).*

Proof. Taking the L^2 -inner product of the equation (2.31) with \mathcal{Y}_t and $\mathcal{Y}_t - \frac{1}{4}\Delta\mathcal{Y} - \Delta\mathcal{Y}_t$, respectively, we get

$$\frac{1}{2} \frac{d}{dt} (\|\mathcal{Y}_t\|_0^2 + \|\partial_3 \mathcal{Y}\|_0^2) + \|\nabla \mathcal{Y}_t\|_0^2 = 0$$

and

$$\frac{d}{dt} \left(\frac{1}{2} \|\mathcal{Y}_t\|_1^2 + \|\partial_3 \mathcal{Y}\|_1^2 + \frac{1}{4} \|\Delta \mathcal{Y}\|_0^2 - \frac{1}{4} (\mathcal{Y}_t | \Delta \mathcal{Y})_{L^2} \right) + \frac{3}{4} \|\nabla \mathcal{Y}_t\|_0^2 + \|\Delta \mathcal{Y}_t\|_0^2 + \frac{1}{4} \|\nabla \partial_3 \mathcal{Y}\|_0^2 = 0.$$

Integrating the above equalities with respect to t gives rise to

$$\|(\mathcal{Y}_t, \partial_3 \mathcal{Y})\|_{L_t^\infty(L^2)} + \|\nabla \partial_t \mathcal{Y}\|_{L_t^2(L^2)} \leq \|(\partial_3 \mathcal{Y}_0, \mathcal{Y}_1)\|_0 \quad \text{and}$$

$$\|(\mathcal{Y}_t, \partial_3 \mathcal{Y})\|_{L_t^\infty(H^1)} + \|\Delta \mathcal{Y}\|_{L_t^\infty(L^2)} + \|\nabla \mathcal{Y}_t\|_{L_t^2(H^1)} + \|\nabla \partial_3 \mathcal{Y}\|_{L_t^2(L^2)} \leq C (\|(\partial_3 \mathcal{Y}_0, \mathcal{Y}_1)\|_1 + \|\Delta \mathcal{Y}_0\|_0).$$

This proves (2.33) for $N = 0$. The general case with $N > 0$ follows similarly.

To show (2.34), we first get, by taking the H^N -inner product of the equation (2.31) with \mathcal{Y}_t , that

$$\frac{1}{2} \frac{d}{dt} (\|\mathcal{Y}_t\|_N^2 + \|\partial_3 \mathcal{Y}\|_N^2) + \|\nabla \mathcal{Y}_t\|_N^2 = 0.$$

So that for any nonnegative $f(t) \in C^1([0, \infty[)$, we have

$$\frac{d}{dt} (f(t) (\|\mathcal{Y}_t\|_N^2 + \|\partial_3 \mathcal{Y}\|_N^2)) + 2f(t) \|\nabla \mathcal{Y}_t\|_N^2 = f'(t) (\|\mathcal{Y}_t\|_N^2 + \|\partial_3 \mathcal{Y}\|_N^2).$$

Taking $f(t) = \langle t \rangle$ and integrating the resulting equality over $[0, t]$, we find

(3.15)

$$\langle t \rangle (\|\mathcal{Y}_t(t)\|_N^2 + \|\partial_3 \mathcal{Y}(t)\|_N^2) + 2 \int_0^t \langle s \rangle \|\nabla \mathcal{Y}_t(s)\|_N^2 ds \leq \|(\partial_3 \mathcal{Y}_0, \mathcal{Y}_1)\|_N^2 + \int_0^t (\|\mathcal{Y}_t\|_N^2 + \|\partial_3 \mathcal{Y}\|_N^2) ds.$$

Yet it from (2.33) that

$$\|\mathcal{Y}_t\|_{L_t^2(H^{N+1})} + \|\partial_3 \mathcal{Y}\|_{L_t^2(H^N)} \leq C_N (\| |D|^{-1} (\partial_3 \mathcal{Y}_0, \mathcal{Y}_1) \|_{N+1} + \|\nabla \mathcal{Y}_0\|_N),$$

which together with (3.15) ensures (2.34). \square

Recall that $\mathcal{Y}(t) = \Gamma(t)Y_1$ is the solution to (2.31) with initial data $(\mathcal{Y}_0, \mathcal{Y}_1) = (0, Y_1)$, so that one can deduce estimates for the operator Γ from the energy estimates (2.33) and (2.34): Indeed combining (3.13) with (2.33) gives

$$(3.16) \quad \|\langle t \rangle \Delta \partial_t \Gamma(t)Y_1\|_{L_t^\infty(H^N)} + \|\langle t \rangle \partial_3^2 \Gamma(t)Y_1\|_{L_t^\infty(H^N)} \leq C_N \|Y_1\|_{N+2}.$$

Let us remark that

$$(3.17) \quad \begin{aligned} \|Y\|_{L^\infty} &\leq \int |\widehat{Y}(\xi)| d\xi \leq \int_{|\xi| \leq 1} |\xi|^{-1} \cdot |\xi| |\widehat{Y}(\xi)| d\xi + \int_{|\xi| > 1} |\xi|^{-2} \cdot |\xi|^2 |\widehat{Y}(\xi)| d\xi \\ &\leq C(\|D|Y\|_{L^2} + \|D|^2 Y\|_{L^2}) \leq C\|D|Y\|_1. \end{aligned}$$

summarizing (2.33), (3.16) and (3.17) then leads to

Corollary 3.1. *For $N \geq 0$, there exists $C_N > 0$ such that*

$$(3.18) \quad \begin{aligned} \|\Gamma(t)Y_1\|_{L_t^\infty(W^{N,\infty})} &\leq C_N \|D|^{-1}Y_1\|_{N+2}, \\ \|\partial_3 \Gamma(t)Y_1\|_{L_t^2(W^{N,\infty})} &\leq C_N \|Y_1\|_{N+2}, \\ \|\langle t \rangle \partial_t \Gamma(t)Y_1\|_{L_t^\infty(W^{N,\infty})} &\leq C_N \|D|^{-1}Y_1\|_{N+3}, \end{aligned}$$

where $\Gamma(t)$ is the solution operator given by (3.1).

Now we are in a position to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. (2.33) and (2.34) are already proved by Lemma 3.2. So it remains to deal with the estimates of (2.32) and (2.35). As a matter of fact, according to the definition of the solution operator $\Gamma(t)$ given by (3.1), we have

$$(3.19) \quad \mathcal{Y}(t) = \partial_t \Gamma(t) \mathcal{Y}_0 + \Gamma(t)(\mathcal{Y}_1 - \Delta \mathcal{Y}_0),$$

from which and (3.2), we infer that for any $\delta \in]0, 1[$ and for $N \in \mathbb{N}$,

$$(3.20) \quad \begin{aligned} |\partial_3 \mathcal{Y}|_{1,N} + |\partial_t \mathcal{Y}|_{\frac{3}{2}-\delta,N} + |\mathcal{Y}|_{\frac{1}{2},N} &\leq |\partial_3 \partial_t \Gamma(t) \mathcal{Y}_0|_{1,N} + |\partial_t^2 \Gamma(t) \mathcal{Y}_0|_{\frac{3}{2}-\delta,N} \\ &\quad + |\partial_t \Gamma(t) \mathcal{Y}_0|_{\frac{1}{2},N} + C_N (\|D|^{2\delta}(\Delta \mathcal{Y}_0, \mathcal{Y}_1)\|_{L^1} + \|D|^{N+4}(\Delta \mathcal{Y}_0, \mathcal{Y}_1)\|_{L^1}). \end{aligned}$$

While notice that $\partial_t^2 \Gamma(t) \mathcal{Y}_0 = \Delta \partial_t \Gamma(t) \mathcal{Y}_0 + \partial_3^2 \Gamma(t) \mathcal{Y}_0$, we get, by applying (3.2) once again, that

$$\begin{aligned} |\partial_3 \partial_t \Gamma(t) \mathcal{Y}_0|_{1,N} &= |\partial_t \Gamma(t) \partial_3 \mathcal{Y}_0|_{1,N} \leq C_N (\|D|^{2\delta} \partial_3 \mathcal{Y}_0\|_{L^1} + \|D|^{N+4} \partial_3 \mathcal{Y}_0\|_{L^1}), \\ |\partial_t^2 \Gamma(t) \mathcal{Y}_0|_{\frac{3}{2}-\delta,N} &\leq |\Delta \partial_t \Gamma(t) \mathcal{Y}_0|_{\frac{3}{2}-\delta,N} + |\partial_3^2 \Gamma(t) \mathcal{Y}_0|_{\frac{3}{2},N} \leq C_N (\|D|^{2\delta} \mathcal{Y}_0\|_{L^1} + \|D|^{N+6} \mathcal{Y}_0\|_{L^1}), \\ |\partial_t \Gamma(t) \mathcal{Y}_0|_{\frac{1}{2},N} &\leq C_N (\|D|^{2\delta} \mathcal{Y}_0\|_{L^1} + \|D|^{N+4} \mathcal{Y}_0\|_{L^1}). \end{aligned}$$

Inserting the above estimates into (3.20) leads to (2.32).

Finally notice that $\Delta \partial_t^2 \Gamma(t) \mathcal{Y}_0 = \Delta^2 \partial_t \Gamma(t) \mathcal{Y}_0 + \Delta \partial_3^2 \Gamma(t) \mathcal{Y}_0$. Then by virtue of (3.16), we deduce

$$\begin{aligned} \|\langle t \rangle \Delta \partial_t \mathcal{Y}(t)\|_{L_t^\infty(H^N)} &\leq \|\langle t \rangle \Delta \partial_t^2 \Gamma(t) \mathcal{Y}_0\|_{L_t^\infty(H^N)} + \|\langle t \rangle \Delta \partial_t \Gamma(t)(\mathcal{Y}_1 - \Delta \mathcal{Y}_0)\|_{L_t^\infty(H^N)} \\ &\leq C_N (\|\Delta \mathcal{Y}_0\|_{N+2} + \|(\Delta \mathcal{Y}_0, \mathcal{Y}_1)\|_{N+2}). \end{aligned}$$

This proves (2.35), and thus we complete the proof of Proposition 2.1. \square

3.3. Decay estimates for the inhomogeneous equation.

Proof of Proposition 2.2. In view of (3.1), we get, by applying Duhamel's principle to (2.36), that

$$(3.21) \quad Y(t) = \int_0^t \Gamma(t-s)g(s)ds.$$

In what follows, we shall present the proof of (2.37) term by term.

• Decay estimate of $\partial_3 Y$.

We first separate the integral in (3.21) as

$$\partial_3 Y(t, y) = \int_0^t \partial_3 \Gamma(t-s)g(s)ds = \int_0^{t/2} \partial_3 \Gamma(t-s)g(s)ds + \int_{t/2}^t \partial_3 \Gamma(t-s)g(s)ds.$$

We deduce from (3.8) that

$$\begin{aligned} \langle t \rangle \int_0^{t/2} |\partial_3 \Gamma(t-s)g(s)|_N ds &\leq C_N \langle t \rangle \int_0^{t/2} \langle t-s \rangle^{-1} |||D|g(s)|||_{N+2} ds \\ &\leq C_N \int_0^{t/2} |||D|g(s)|||_{N+2} ds \leq C |||D|g|||_{L_t^1(W^{N+2,1})}. \end{aligned}$$

While it follows from the second inequality in (3.18) that

$$\langle t \rangle \int_{t/2}^t |\partial_3 \Gamma(t-s)g(s)|_N ds \leq \langle t \rangle \left(\int_{t/2}^t \|g(s)\|_{N+2}^2 ds \right)^{\frac{1}{2}} \leq C \theta^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} g\|_{L_t^2(H^{N+2})}.$$

Hence we achieve

$$(3.22) \quad |\partial_3 Y|_{1,N} \leq C_N \left(|||D|g|||_{L_t^1(W^{N+2,1})} + \theta^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} g\|_{L_t^2(H^{N+2})} \right).$$

• Decay estimate of Y_t .

Noticing that $\Gamma(0) = 0$, we have

$$Y_t(t) = \int_0^t \partial_t \Gamma(t-s)g(s)ds = \int_0^{t/2} \partial_t \Gamma(t-s)g(s)ds + \int_{t/2}^t \partial_t \Gamma(t-s)g(s)ds.$$

It follows from (3.11) that

$$\begin{aligned} \langle t \rangle^{\frac{3}{2}-\delta} \int_0^{t/2} |\partial_t \Gamma(t-s)g(s)|_N ds &\leq C_N \langle t \rangle^{\frac{3}{2}-\delta} \int_0^{t/2} \langle t-s \rangle^{-(\frac{3}{2}-\delta)} (|||D|^{2\delta}g(s)|||_N + |||D^4g(s)|||_N) ds \\ &\leq C_N (|||D|^{2\delta}g|||_{L_t^1(W^{N,1})} + |||D^4g|||_{L_t^2(W^{N,1})}). \end{aligned}$$

Whereas it follows from the third inequality in (3.18) that

$$\begin{aligned} \langle t \rangle^{\frac{3}{2}-\delta} \int_{t/2}^t |\partial_t \Gamma(t-s)g(s)|_N ds &\leq \int_{t/2}^t \langle t-s \rangle^{-1} |||D|^{-1}g(s)|||_{N+3} \langle s \rangle^{\frac{3}{2}-\delta} ds \\ &\leq C_N \log \langle \theta \rangle \|\langle t \rangle^{\frac{3}{2}-\delta} |D|^{-1}g\|_{L_t^\infty(H^{N+3})} \leq C_N \log \langle \theta \rangle |||D|^{-1}g|||_{\frac{3}{2}-\delta, N+3}. \end{aligned}$$

As a result, it comes out

$$(3.23) \quad |Y_t|_{\frac{3}{2}-\delta, N} \leq C_N \left((|||D|^{2\delta}g|||_{L_t^1(W^{N,1})} + |||D|^4g|||_{L_t^1(W^{N,1})}) + \log \langle \theta \rangle |||D|^{-1}g|||_{\frac{3}{2}-\delta, N+3} \right).$$

• Decay estimate of Y .

As in the previous steps, we first split the integral (3.21) into two parts. For the integral from 0 to $t/2$, we use (3.12) to deduce that

$$\begin{aligned} \langle t \rangle^{\frac{1}{2}} \int_0^{t/2} |\Gamma(t-s)g(s)|_N ds &\leq \langle t \rangle^{\frac{1}{2}} \int_0^{t/2} C_N \langle t-s \rangle^{-\frac{1}{2}} (|||D|^{\frac{1}{2}}g(s)|||_N + |||D|^2g(s)|||_N) ds \\ &\leq C_N \left(|||D|^{\frac{1}{2}}g|||_{L_t^1(W^{N,1})} + |||D|^2g|||_{L_t^1(W^{N,1})} \right). \end{aligned}$$

For the integral from $t/2$ to t , we apply the first inequality of (3.18) to get

$$\begin{aligned} \langle t \rangle^{\frac{1}{2}} \int_{t/2}^t |\Gamma(t-s)g(s)|_N ds &\leq C_N \langle t \rangle^{\frac{1}{2}} \int_{t/2}^t |||D|^{-1}g(s)|||_{N+2} ds \\ &\leq C_N \langle t \rangle \left(\int_{t/2}^t |||D|^{-1}g(s)|||_{N+2}^2 ds \right)^{\frac{1}{2}} \leq C_N \theta^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1}g\|_{L_t^2(H^{N+2})}. \end{aligned}$$

Hence we obtain

$$(3.24) \quad |Y|_{\frac{1}{2}, N} \leq C_N \left((|||D|^{\frac{1}{2}}g|||_{L_t^1(W^{N,1})} + |||D|^2g|||_{L_t^1(W^{N,1})}) + \theta^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1}g\|_{L_t^2(H^{N+2})} \right).$$

By summarizing the estimates (3.22), (3.23) and (3.24), we complete the proof of (2.37). \square

4. THE DERIVATIVES OF f GIVEN BY (2.25)

4.1. Computation of $f'(Y; X)$. The goal of this subsection is to derive the linearized equations of the system (2.1-2.25). We first decompose the pressure function \mathbf{p} given by (2.25) as $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ with

$$(4.1) \quad \mathbf{p}_1 \stackrel{\text{def}}{=} -\Delta^{-1} \operatorname{div}((\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}_1) + \Delta^{-1} \operatorname{div}(\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)))$$

$$(4.2) \quad \mathbf{p}_2 \stackrel{\text{def}}{=} -\Delta^{-1} \operatorname{div}((\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}_2) + \Delta^{-1} \operatorname{div}(\mathcal{A} \operatorname{div}(\mathcal{A}(Y_t \otimes Y_t))).$$

Let us denote

$$(4.3) \quad f_0 \stackrel{\text{def}}{=} \nabla_y \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla_y Y_t), \quad f_1 \stackrel{\text{def}}{=} \mathcal{A}^t \nabla_y \mathbf{p}_1 \quad \text{and} \quad f_2 \stackrel{\text{def}}{=} \mathcal{A}^t \nabla_y \mathbf{p}_2.$$

Then the functional f given by (2.25) can be decomposed as $f_0 - f_1 + f_2$.

Before proceeding, let us recall that for a map $f : \mathcal{U} \rightarrow \mathcal{Y}$, where \mathcal{U} is an open set of \mathcal{X} and $\mathcal{X} \stackrel{\text{def}}{=} C^\infty([0, \infty[\times \mathbb{R}^3; \mathbb{R}^3)$, the differentiation of f at $Y \in \mathcal{U}$ along the direction $X \in \mathcal{X}$ is defined as

$$f'(Y; X) \stackrel{\text{def}}{=} \lim_{s \rightarrow 0} \frac{f(Y + sX) - f(Y)}{s} = \frac{d}{ds} f(Y + sX)|_{s=0}.$$

For $f \in C^\infty([0, \infty) \times \mathbb{R}^3; M_{3 \times 3}(\mathbb{R}))$, $g \in C^\infty([0, \infty) \times \mathbb{R}^3; \mathbb{R}^3)$, we have

$$(fg)'(Y; X) = f'(Y; X)g(Y) + f(Y)g'(Y; X).$$

Then for $\mathcal{A}(Y) = (Id + \nabla Y)^{-1}$, we have

$$(4.4) \quad \mathcal{A}'(Y; X) = \mathcal{A}(-\nabla X)\mathcal{A}, \quad \text{and} \quad (\mathcal{A}^t)'(Y; X) = \mathcal{A}^t(-\nabla X)^t \mathcal{A}^t$$

and thus

$$(4.5) \quad (\mathcal{A}\mathcal{A}^t - Id)'(Y; X) = \mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(-\nabla X)^t \mathcal{A}^t.$$

As a result, we deduce that

$$(4.6) \quad \begin{aligned} f'_0(Y; X) &= \nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\nabla Y_t) + \nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t) \\ &= \nabla \cdot ((\mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(-\nabla X)^t \mathcal{A}^t)\nabla Y_t) + \nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t). \end{aligned}$$

For $m = 1, 2$, we have

$$(4.7) \quad \begin{aligned} f'_m(Y; X) &= (\mathcal{A}^t)'(Y; X)\nabla \mathbf{p}_m(Y) + \mathcal{A}^t \nabla \mathbf{p}'_m(Y; X) \\ &= -\mathcal{A}^t(\nabla X)^t \mathcal{A}^t(\nabla \mathbf{p}_m)(Y) + \mathcal{A}^t \nabla \mathbf{p}'_m(Y; X). \end{aligned}$$

Moreover, it follows from (4.1) that

$$(4.8) \quad \begin{aligned} \mathbf{p}'_1(Y; X) &= \Delta^{-1} \operatorname{div} \left(-(\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}'_1(Y; X) - (\mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(-\nabla X)^t \mathcal{A}^t)\nabla \mathbf{p}_1(Y) \right. \\ &\quad \left. + \mathcal{A} \operatorname{div}((\mathcal{A}(-\nabla X)\mathcal{A})(\partial_3 Y \otimes \partial_3 Y)) + (\mathcal{A}(-\nabla X)\mathcal{A}) \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)) \right. \\ &\quad \left. + \mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)) \right). \end{aligned}$$

And similarly, it follows from (4.2) that

$$(4.9) \quad \begin{aligned} \mathbf{p}'_2(Y; X) &= \Delta^{-1} \operatorname{div} \left(-(\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}'_2(Y; X) - (\mathcal{A}(-\nabla X)\mathcal{A}\mathcal{A}^t + \mathcal{A}\mathcal{A}^t(-\nabla X)^t \mathcal{A}^t)\nabla \mathbf{p}_2(Y) \right. \\ &\quad \left. + \mathcal{A} \operatorname{div}((\mathcal{A}(-\nabla X)\mathcal{A})(Y_t \otimes Y_t)) + (\mathcal{A}(-\nabla X)\mathcal{A}) \operatorname{div}(\mathcal{A}(Y_t \otimes Y_t)) \right. \\ &\quad \left. + \mathcal{A} \operatorname{div}(\mathcal{A}(Y_t \otimes X_t + X_t \otimes Y_t)) \right). \end{aligned}$$

The linearized equation of (2.1-2.25) then reads as (2.40).

Remark 4.1. Let $V \in C^\infty([0, +\infty) \times \mathbb{R}^3; M_{3 \times 3}(\mathbb{R}))$ and $U \in C^\infty([0, +\infty) \times \mathbb{R}^3; \mathbb{R}^3)$, we denote $h(V) \stackrel{\text{def}}{=} (Id + V)^{-1}$, and

$$F_0(U, V) \stackrel{\text{def}}{=} \nabla \cdot \left((h(V)h(V)^t - Id)U \right), \quad F(U, V) \stackrel{\text{def}}{=} h(V)^t \mathbf{q}(U, V) \quad \text{with} \\ \mathbf{q} \stackrel{\text{def}}{=} -\Delta^{-1} \operatorname{div}((h(V)h(V)^t - Id)\nabla \mathbf{q}) + \Delta^{-1} \operatorname{div}(h(V) \operatorname{div}(h(V)(U \otimes U))).$$

Then f_0, f_1, f_2 defined by (4.3) can be written as

$$f_0 = F_0(\nabla Y_t, \nabla Y), \quad f_1 = F(\partial_3 Y, \nabla Y) \quad \text{and} \quad f_2 = F(Y_t, \nabla Y),$$

and hence f'_0, f'_1 and f'_2 read

$$f'_0(Y; X) = F'_{0,U}(\nabla Y_t, \nabla Y) \nabla X_t + F'_{0,V}(\nabla Y_t, \nabla Y) \nabla X, \\ f'_1(Y; X) = F'_U(\partial_3 Y, \nabla Y) \partial_3 X + F'_V(\partial_3 Y, \nabla Y) \nabla X, \\ f'_2(Y; X) = F'_U(Y_t, \nabla Y) X_t + F'_V(Y_t, \nabla Y) \nabla X,$$

where the functionals $F'_{0,U}(U, V)$, $F'_{0,V}(U, V)$, $F'_U(U, V)$ and $F'_V(U, V)$ are given by

$$F'_{0,U}(U, V) \dot{U} = \nabla \cdot \left((h(V)h(V)^t - Id) \dot{U} \right), \\ F'_{0,V}(U, V) \dot{V} = \nabla \cdot \left(((h'(V)\dot{V})h(V)^t + h(V)(h'(V)\dot{V})^t)U \right), \\ F'_U(U, V) \dot{U} = h(V)^t \mathbf{q}'_U(U, V) \dot{U} \quad \text{and} \quad F'_V(U, V) \dot{V} = (h'(V)\dot{V})^t \mathbf{q}(U, V) + h(V)^t \mathbf{q}'_V(U, V) \dot{V},$$

and

$$h'(V)\dot{V} = (Id + V)^{-1}(-\dot{V})(Id + V)^{-1}, \\ \mathbf{q}'_U(U, V) \dot{U} = -\Delta^{-1} \operatorname{div} \left((h(V)h(V)^t - Id) \nabla \mathbf{q}'_U(U, V) \dot{U} - h(V) \operatorname{div}(h(V)(U \otimes \dot{U} + \dot{U} \otimes U)) \right) \\ \mathbf{q}'_V(U, V) \dot{V} = -\Delta^{-1} \operatorname{div} \left((h(V)h(V)^t - Id) \nabla \mathbf{q}'_V(U, V) \dot{V} - (h'(V)\dot{V}) \operatorname{div}(h(V)(U \otimes U)) \right. \\ \left. - h(V) \operatorname{div}((h'(V)\dot{V})(U \otimes U)) + ((h'(V)\dot{V})h(V)^t + h(V)(h'(V)\dot{V})^t) \nabla \mathbf{q} \right).$$

4.2. Computation of $f''(Y; X, W)$. In order to estimate the error arisen in the Nash-Moser iteration scheme, we need the second derivatives of f . Toward this, let us recall the product rule

$$(fg)''(Y; X, W) = f''(Y; X, W)g(Y) + f(Y)g''(Y; X, W) \\ (4.10) \quad + f'(Y; X)g'(Y; W) + f'(Y; W)g'(Y; X).$$

It is easy to observe from (4.4) that

$$(4.11) \quad \mathcal{A}''(Y; X, W) = \mathcal{A}(\nabla X) \mathcal{A}(\nabla W) \mathcal{A} + \mathcal{A}(\nabla W) \mathcal{A}(\nabla X) \mathcal{A}.$$

Then applying the product rule (4.10) and (4.4) gives

$$(\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W) = \mathcal{A}(\nabla X) \mathcal{A}(\nabla W) \mathcal{A}\mathcal{A}^t + \mathcal{A}(\nabla W) \mathcal{A}(\nabla X) \mathcal{A}\mathcal{A}^t \\ (4.12) \quad + \mathcal{A}\mathcal{A}^t(\nabla X)^t \mathcal{A}^t(\nabla W)^t \mathcal{A}^t + \mathcal{A}\mathcal{A}^t(\nabla W)^t \mathcal{A}^t(\nabla X)^t \mathcal{A}^t \\ + \mathcal{A}(\nabla X) \mathcal{A}\mathcal{A}^t(\nabla W)^t \mathcal{A}^t + \mathcal{A}(\nabla W) \mathcal{A}\mathcal{A}^t(\nabla X)^t \mathcal{A}^t.$$

Recall that f_0 is given by (4.3), we deduce from (4.10) that

$$(4.13) \quad f''_0(Y; X, W) = \nabla \cdot \left((\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W) \nabla Y_t \right) \\ + \nabla \cdot \left((\mathcal{A}\mathcal{A}^t - Id)'(Y; X) \nabla W_t \right) + \nabla \cdot \left((\mathcal{A}\mathcal{A}^t - Id)'(Y; W) \nabla X_t \right).$$

Similarly for $f_m(Y) = \mathcal{A}^t \nabla \mathbf{p}_m$, $m = 1, 2$, we have

$$(4.14) \quad \begin{aligned} f_m''(Y; X, W) &= (\mathcal{A}^t)''(Y; X, W) \nabla \mathbf{p}_m(Y) + \mathcal{A}^t \nabla (\mathbf{p}_m''(Y; X, W)) \\ &\quad + (\mathcal{A}^t)'(Y; X) \nabla (\mathbf{p}_m'(Y; W)) + (\mathcal{A}^t)'(Y; W) \nabla (\mathbf{p}_m'(Y; X)). \end{aligned}$$

Then in view of (4.8), (4.9), to obtain the expression of $f_m''(Y; X, W)$, $m = 1, 2$, it remains to calculate $\mathbf{p}_m''(Y; X, W)$, $m = 1, 2$. Indeed, it follows from (4.1), (4.2) and (4.10) that

$$(4.15) \quad \begin{aligned} \mathbf{p}_1''(Y; X, W) &= -\Delta^{-1} \operatorname{div} \left((\mathcal{A}\mathcal{A}^t - Id) \nabla \mathbf{p}_1''(Y; X, W) + (\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W) \nabla \mathbf{p}_1(Y) \right) \\ &\quad + (\mathcal{A}\mathcal{A}^t - Id)'(Y; X) \nabla \mathbf{p}_1'(Y; W) + (\mathcal{A}\mathcal{A}^t - Id)'(Y; W) \nabla \mathbf{p}_1'(Y; X) \\ &\quad - \mathcal{A} \operatorname{div} [\mathcal{A}(\partial_3 X \otimes \partial_3 W + \partial_3 W \otimes \partial_3 X) + \mathcal{A}''(Y; X, W)(\partial_3 Y \otimes \partial_3 Y)] \\ &\quad + \mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 W + \partial_3 W \otimes \partial_3 Y) + \mathcal{A}'(Y; W)(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y) \\ &\quad - \mathcal{A}'(Y; X) \operatorname{div} [\mathcal{A}'(Y; W)(\partial_3 Y \otimes \partial_3 Y) + \mathcal{A}(\partial_3 Y \otimes \partial_3 W + \partial_3 W \otimes \partial_3 Y)] \\ &\quad - \mathcal{A}'(Y; W) \operatorname{div} [\mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 Y) + \mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)] \\ &\quad - \mathcal{A}''(Y; X, W) \operatorname{div} (\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)). \end{aligned}$$

and

$$(4.16) \quad \begin{aligned} \mathbf{p}_2''(Y; X, W) &= -\Delta^{-1} \operatorname{div} \left((\mathcal{A}\mathcal{A}^t - Id) \nabla \mathbf{p}_2''(Y; X, W) + (\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W) \nabla \mathbf{p}_2(Y) \right) \\ &\quad + (\mathcal{A}\mathcal{A}^t - Id)'(Y; X) \nabla \mathbf{p}_2'(Y; W) + (\mathcal{A}\mathcal{A}^t - Id)'(Y; W) \nabla \mathbf{p}_2'(Y; X) \\ &\quad - \mathcal{A} \operatorname{div} [\mathcal{A}(X_t \otimes W_t + W_t \otimes X_t) + \mathcal{A}''(Y; X, W)(Y_t \otimes Y_t)] \\ &\quad + \mathcal{A}'(Y; X)(Y_t \otimes W_t + W_t \otimes Y_t) + \mathcal{A}'(Y; W)(Y_t \otimes X_t + X_t \otimes Y_t) \\ &\quad - \mathcal{A}'(Y; X) \operatorname{div} [\mathcal{A}(Y_t \otimes W_t + W_t \otimes Y_t) + \mathcal{A}'(Y; W)(Y_t \otimes Y_t)] \\ &\quad - \mathcal{A}'(Y; W) \operatorname{div} [\mathcal{A}(Y_t \otimes X_t + X_t \otimes Y_t) + \mathcal{A}'(Y; X)(Y_t \otimes Y_t)] \\ &\quad - \mathcal{A}''(Y; X, W) \operatorname{div} (\mathcal{A}(Y_t \otimes Y_t)). \end{aligned}$$

Remark 4.2. In view of Remark 4.1, f_m'' can be written as follows

$$\begin{aligned} f_0''(Y; X, W) &= F_{0,UU}''(\nabla Y_t, \nabla Y) \nabla X_t \cdot \nabla W_t + F_{0,UV}''(\nabla Y_t, \nabla Y) \nabla X_t \cdot \nabla W \\ &\quad + F_{0,VU}''(\nabla Y_t, \nabla Y) \nabla W_t \cdot \nabla X + F_{0,VV}''(\nabla Y_t, \nabla Y) \nabla X \cdot \nabla W, \\ f_1''(Y; X, W) &= F_{UU}''(\partial_3 Y, \nabla Y) \partial_3 X \cdot \partial_3 W + F_{UV}''(\partial_3 Y, \nabla Y) \partial_3 X \cdot \nabla W \\ &\quad + F_{VU}''(\partial_3 Y, \nabla Y) \nabla X \cdot \partial_3 W + F_{VV}''(\partial_3 Y, \nabla Y) \nabla X \cdot \nabla W, \\ f_2''(Y; X, W) &= F_{UU}''(Y_t, \nabla Y) X_t \cdot W_t + F_{UV}''(Y_t, \nabla Y) X_t \cdot \nabla W \\ &\quad + F_{VU}''(Y_t, \nabla Y) \nabla X \cdot W_t + F_{VV}''(Y_t, \nabla Y) \nabla X \cdot \nabla W, \end{aligned}$$

where $F_{0,UU}''(U, V) \dot{U}_1 \cdot \dot{U}_2 = 0$,

$$\begin{aligned} F_{0,UV}''(U, V) \dot{U} \cdot \dot{V} &= \nabla \cdot \left(((h'(V) \dot{V}) h(V)^t + h(V) (h'(V) \dot{V})^t) \dot{U} \right) = F_{0,VU}''(U, V) \dot{V} \cdot \dot{U}, \\ F_{0,VV}''(U, V) \dot{V}_1 \cdot \dot{V}_2 &= \nabla \cdot \left(((h''(V) \dot{V}_1 \cdot \dot{V}_2) h(V)^t + h(V) (h''(V) \dot{V}_1 \cdot \dot{V}_2)^t + \right. \\ &\quad \left. + ((h'(V) \dot{V}_1) (h'(V) \dot{V}_2)^t + (h'(V) \dot{V}_2) (h'(V) \dot{V}_1)^t) U \right) \quad \text{with} \\ h''(V) \dot{V}_1 \cdot \dot{V}_2 &= (Id + V)^{-1} (-\dot{V}_1) (Id + V)^{-1} (-\dot{V}_2) (Id + V)^{-1} \\ &\quad + (Id + V)^{-1} (-\dot{V}_2) (Id + V)^{-1} (-\dot{V}_1) (Id + V)^{-1}; \end{aligned}$$

and

$$\begin{aligned}
F''_{UU}(U, V)\dot{U}_1 \cdot \dot{U}_2 &= h(V)^t \mathbf{q}''_{UU}(U, V)\dot{U}_1 \cdot \dot{U}_2, \\
F''_{UV}(U, V)\dot{U} \cdot \dot{V} &= h(V)^t \mathbf{q}''_{UV}(U, V)\dot{U} \cdot \dot{V} + (h'(V)\dot{V})^t \mathbf{q}'_U(U, V)\dot{U} = F''_{VU}(U, V)\dot{V} \cdot \dot{U}, \\
F''_{VV}(U, V)\dot{V}_1 \cdot \dot{V}_2 &= (h''(V)\dot{V}_1 \cdot \dot{V}_2)^t \mathbf{q}(U, V) + h(V)^t \mathbf{q}''_{VV}(U, V)\dot{V}_1 \cdot \dot{V}_2 \\
&\quad + (h'(V)\dot{V}_1)^t \mathbf{q}'_V(U, V)\dot{V}_2 + (h'(V)\dot{V}_2)^t \mathbf{q}'_V(U, V)\dot{V}_1,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{q}''_{UU}(U, V)\dot{U}_1 \cdot \dot{U}_2 &= -\Delta^{-1} \operatorname{div} \left((h(V)h(V)^t - Id) \nabla \mathbf{q}''_{UU}(U, V)\dot{U}_1 \cdot \dot{U}_2 \right. \\
&\quad \left. - h(V) \operatorname{div}(h(V)(\dot{U}_1 \otimes \dot{U}_2 + \dot{U}_2 \otimes \dot{U}_1)) \right),
\end{aligned}$$

$$\begin{aligned}
\mathbf{q}''_{UV}(U, V)\dot{U} \cdot \dot{V} &= -\Delta^{-1} \operatorname{div} \left((h(V)h(V)^t - Id) \nabla \mathbf{q}''_{UV}(U, V)\dot{U} \cdot \dot{V} \right. \\
&\quad + ((h'(V)\dot{V})h(V)^t + h(V)(h'(V)\dot{V})^t) \nabla \mathbf{q}'_U(U, V)\dot{U} - (h'(V)\dot{V}) \operatorname{div}(h(V)(\dot{U} \otimes U + U \otimes \dot{U})) \\
&\quad \left. - h(V) \operatorname{div}((h'(V)\dot{V})(\dot{U} \otimes U + U \otimes \dot{U})) \right) = \mathbf{q}''_{VU}(U, V)\dot{V} \cdot \dot{U},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{q}''_{VV}(U, V)\dot{V}_1 \cdot \dot{V}_2 &= -\Delta^{-1} \operatorname{div} \left(((h'(V)\dot{V}_2)h(V)^t + h(V)(h'(V)\dot{V}_2)^t) \nabla \mathbf{q}'_V(U, V)\dot{V}_1 \right. \\
&\quad + (h(V)h(V)^t - Id) \nabla \mathbf{q}''_{VV}(U, V)\dot{V}_1 \cdot \dot{V}_2 - (h''(V)\dot{V}_1 \cdot \dot{V}_2) \operatorname{div}(h(V)(U \otimes U)) \\
&\quad + ((h'(V)\dot{V}_1)h(V)^t + h(V)(h'(V)\dot{V}_1)^t) \nabla \mathbf{q}'_V(U, V)\dot{V}_2 - (h'(V)\dot{V}_1) \operatorname{div}((h'(V)\dot{V}_2)(U \otimes U)) \\
&\quad + ((h''(V)\dot{V}_1 \cdot \dot{V}_2)h(V)^t + h(V)(h''(V)\dot{V}_1 \cdot \dot{V}_2)^t - (h'(V)\dot{V}_2) \operatorname{div}((h'(V)\dot{V}_1)(U \otimes U)) \\
&\quad \left. + (h'(V)\dot{V}_1)(h'(V)\dot{V}_2)^t + (h'(V)\dot{V}_2)(h'(V)\dot{V}_1)^t) \nabla \mathbf{q} - h(V) \operatorname{div}((h''(V)\dot{V}_1 \cdot \dot{V}_2)(U \otimes U)) \right).
\end{aligned}$$

5. THE ESTIMATES OF $f'(Y; X)$

5.1. The estimate of $\|f'(Y; X)\|_{\delta, N}$. The main result of this subsection is listed in Proposition 2.3. As we explained in the Section 2, the main idea is to use the norm of the homogeneous Besov spaces $\dot{B}_{1,1}^s$ to replace the norm of the classical Sobolev spaces $\dot{W}^{s,1}$. In order to do so, we need not only the product law (2.48) but also the following one.

Lemma 5.1. *For any $s > 0$, there holds*

$$(5.1) \quad \|ab\|_{\dot{B}_{1,1}^s} \leq C \left(\min(\|a\|_0 \|b\|_{\dot{B}_{1,1}^s}, \|a\|_0 \|b\|_{\dot{B}_{2,1}^s}) + \|a\|_{\dot{B}_{2,1}^s} \|b\|_0 \right).$$

Proof. We first get, by applying Bony's decomposition [4] that

$$ab = T_a b + R'(a, b) \quad \text{with}$$

$$T_a b = \sum_{j \in \mathbb{Z}} S_{j-1} a \Delta_j b \quad \text{and} \quad R'(a, b) = \sum_{j \in \mathbb{Z}} \Delta_j a S_{j+2} b.$$

Due to the support properties to the Fourier transform of the terms in $T_a b$, we have

$$\|\dot{\Delta}_j(T_a b)\|_{L^1} \leq \sum_{|j'-j| \leq 4} |S_{j'-1} a|_0 \|\dot{\Delta}_{j'} b\|_{L^1} \lesssim d_j 2^{-js} |a|_0 \|b\|_{\dot{B}_{1,1}^s},$$

where $(d_j)_{j \in \mathbb{Z}}$ is a non-negative generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{j \in \mathbb{Z}} d_j = 1$.

Along the same line, we also have

$$\|\dot{\Delta}_j(T_a b)\|_{L^1} \leq \sum_{|j'-j| \leq 4} \|S_{j'-1} a\|_0 \|\dot{\Delta}_{j'} b\|_0 \lesssim d_j 2^{-js} \|a\|_0 \|b\|_{\dot{B}_{2,1}^s},$$

and

$$\begin{aligned} \|\dot{\Delta}_j(R'(a, b))\|_{L^1} &\leq \sum_{j' \geq j-N_0} \|\dot{\Delta}_{j'} a\|_0 \|S_{j'+2} b\|_0 \\ &\leq \sum_{j' \geq j-N_0} d_{j'} 2^{-j's} \|a\|_{\dot{B}_{2,1}^s} \|b\|_0 \lesssim d_j 2^{-js} \|a\|_{\dot{B}_{2,1}^s} \|b\|_0, \end{aligned}$$

where in the last step, we used the fact that $s > 0$. By summing up the above inequalities, we arrive at (5.1). \square

Notice that $\mathcal{A}(\nabla Y) = (Id + \nabla Y)^{-1}$, we write

$$\mathcal{A}\mathcal{A}^t - Id = (\mathcal{A} - Id)(\mathcal{A} - Id)^t + (\mathcal{A} - Id) + (\mathcal{A} - Id)^t, \quad \mathcal{A} - Id = \sum_{n=1}^{\infty} (-1)^n (\nabla Y)^n.$$

So that under the assumption of (2.41), for $s > 0$, we get, by applying (2.48), that

$$\begin{aligned} \|\mathcal{A}f\|_{\dot{B}_{p,r}^s} &\lesssim (1 + |\mathcal{A} - Id|_0) \|f\|_{\dot{B}_{p,r}^s} + \|\mathcal{A} - Id\|_{\dot{B}_{p,r}^s} |f|_0 \\ (5.2) \quad &\lesssim \|f\|_{\dot{B}_{p,r}^s} + \|\nabla Y\|_{\dot{B}_{p,r}^s} |f|_0, \end{aligned}$$

Along the same line, we get, by applying (5.1), that

$$(5.3) \quad \|\mathcal{A}f\|_{\dot{B}_{1,1}^s} \lesssim \|f\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|f\|_0,$$

5.1.1. *Estimate of $\|f'_0(Y; X)\|_{\dot{B}_{1,1}^s}$.* In view of (4.6), we have

$$\|f'_0(Y; X)\|_{\dot{B}_{1,1}^s} \leq \|\mathcal{A}(\nabla X \mathcal{A} + \mathcal{A}^t(\nabla X)^t) \mathcal{A}^t \nabla Y_t\|_{\dot{B}_{1,1}^{s+1}} + \|(\mathcal{A}\mathcal{A}^t - Id) \nabla X_t\|_{\dot{B}_{1,1}^{s+1}}.$$

It follows from (2.41) and (5.1) that

$$\|(\mathcal{A}\mathcal{A}^t - Id) \nabla X_t\|_{\dot{B}_{1,1}^{s+1}} \lesssim \|\nabla Y\|_0 \|\nabla X_t\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} \|\nabla X_t\|_0.$$

While applying (5.3) gives

$$\|\mathcal{A} \nabla X \mathcal{A}^t \nabla Y_t\|_{\dot{B}_{1,1}^{s+1}} \lesssim \|\nabla X \mathcal{A}^t \nabla Y_t\|_{\dot{B}_{1,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} \|\nabla X \mathcal{A}^t \nabla Y_t\|_0,$$

yet it follows from (2.48) and (5.1) that

$$\begin{aligned} \|\nabla X \mathcal{A}^t \nabla Y_t\|_{\dot{B}_{1,1}^{s+1}} &\lesssim \|\nabla X\|_0 \|\mathcal{A}^t \nabla Y_t\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} \|\mathcal{A}^t \nabla Y_t\|_0 \\ &\lesssim \|\nabla X\|_0 (\|\nabla Y_t\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} |\nabla Y_t|_0) + \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} \|\nabla Y_t\|_0, \end{aligned}$$

so that there holds

$$\|\mathcal{A} \nabla X \mathcal{A}^t \nabla Y_t\|_{\dot{B}_{1,1}^{s+1}} \lesssim \|\nabla Y_t\|_0 \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y_t\|_{\dot{B}_{2,1}^{s+1}} + |\nabla Y_t|_0 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|\nabla X\|_0.$$

The same estimate holds for $\|\mathcal{A}\mathcal{A}^t(-\nabla X)^t \mathcal{A}^t \nabla Y_t\|_{\dot{B}_{1,1}^{s+1}}$. As a result, we obtain

$$\begin{aligned} (5.4) \quad \|f'_0(Y; X)\|_{\dot{B}_{1,1}^s} &\leq \|\nabla Y\|_0 \|\nabla X_t\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} \|\nabla X_t\|_0 \\ &\quad + \|\nabla Y_t\|_0 \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y_t\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} |\nabla Y_t|_0) \|\nabla X\|_0. \end{aligned}$$

5.1.2. *Estimate of $\|f'_m(Y; X)\|_{\dot{B}_{1,1}^s}$, $m = 1, 2$.* In view of (4.7), we have

$$\|f'_m(Y; X)\|_{\dot{B}_{1,1}^s} \leq \|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{B}_{1,1}^s} + \|\mathcal{A}^t \nabla(\mathbf{p}'_m(Y; X))\|_{\dot{B}_{1,1}^s}.$$

Applying (5.3) gives

$$\begin{aligned} \|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{B}_{1,1}^s} &\lesssim \|(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_0, \\ \|\mathcal{A}^t \nabla(\mathbf{p}'_m(Y; X))\|_{\dot{B}_{1,1}^s} &\lesssim \|\nabla(\mathbf{p}'_m(Y; X))\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla(\mathbf{p}'_m(Y; X))\|_0. \end{aligned}$$

While applying (2.48) and (5.1) leads to

$$\|(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{B}_{1,1}^s} \lesssim \|\nabla X\|_0 (\|\nabla \mathbf{p}_m\|_{\dot{B}_{2,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} |\nabla \mathbf{p}_m|_0) + \|\nabla X\|_{\dot{B}_{2,1}^s} \|\nabla \mathbf{p}_m\|_0,$$

which yields

$$\|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{B}_{1,1}^s} \lesssim \|\nabla \mathbf{p}_m\|_0 \|\nabla X\|_{\dot{B}_{2,1}^s} + (\|\nabla \mathbf{p}_m\|_{\dot{B}_{2,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} |\nabla \mathbf{p}_m|_0) \|\nabla X\|_0.$$

Hence it comes out

$$\begin{aligned} \|f'_m(Y; X)\|_{\dot{B}_{1,1}^s} &\lesssim \|\nabla \mathbf{p}_m\|_0 \|\nabla X\|_{\dot{B}_{2,1}^s} + (\|\nabla \mathbf{p}_m\|_{\dot{B}_{2,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} |\nabla \mathbf{p}_m|_0) \|\nabla X\|_0 \\ (5.5) \quad &+ \|\nabla(\mathbf{p}'_m(Y; X))\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla(\mathbf{p}'_m(Y; X))\|_0. \end{aligned}$$

It remains to handle the estimates of

$$\|\nabla \mathbf{p}_m\|_0, \quad \|\nabla \mathbf{p}_m\|_{\dot{B}_{2,1}^s}, \quad \|\nabla(\mathbf{p}'_m(Y; X))\|_{\dot{B}_{1,1}^s} \quad \text{and} \quad \|\nabla(\mathbf{p}'_m(Y; X))\|_0.$$

• Estimate of $\|\nabla \mathbf{p}_m\|_0$.

We first deduce from (4.1) that

$$\begin{aligned} \|\nabla \mathbf{p}_1\|_0 &\leq |\mathcal{A} \mathcal{A}^t - Id|_0 \|\nabla \mathbf{p}_1\|_0 + |\mathcal{A}|_0 \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{H}^1} \\ &\leq |\mathcal{A} \mathcal{A}^t - Id|_0 \|\nabla \mathbf{p}_1\|_0 + |\mathcal{A}|_0 (1 + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \|\partial_3 Y \otimes \partial_3 Y\|_{\dot{H}^1}. \end{aligned}$$

Due to the assumption (2.41), one has

$$|\mathcal{A} \mathcal{A}^t - Id|_0 \lesssim |\nabla Y|_0 \lesssim \|\nabla Y\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \leq \delta_1,$$

so that we infer

$$(5.6) \quad \|\nabla \mathbf{p}_1\|_0 \lesssim \|\partial_3 Y \otimes \partial_3 Y\|_{\dot{H}^1} \lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_1.$$

Similarly, we have

$$(5.7) \quad \|\nabla \mathbf{p}_2\|_0 \lesssim |Y_t|_0 \|Y_t\|_1.$$

• Estimates of $\|\nabla \mathbf{p}_m\|_{\dot{B}_{2,1}^s}$ for $s > 0$.

We start with the estimate of $\|\nabla \mathbf{p}_m\|_{\dot{B}_{2,1}^{\frac{3}{2}}}$. Indeed by (4.1), one has

$$\|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \lesssim \|\mathcal{A} \mathcal{A}^t - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{2,1}^{\frac{3}{2}}},$$

from which, (2.41) and the product law (2.48), we infer

$$\begin{aligned} \|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} &\lesssim (1 + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{B}_{2,1}^{\frac{5}{2}}} \\ &\lesssim (1 + |\mathcal{A} - Id|_0) \|\partial_3 Y \otimes \partial_3 Y\|_{\dot{B}_{2,1}^{\frac{5}{2}}} + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^{\frac{5}{2}}} |\partial_3 Y \otimes \partial_3 Y|_0. \end{aligned}$$

As a result, by virtue of (2.41), it comes out

$$(5.8) \quad \|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \lesssim |\partial_3 Y|_0 (\|\partial_3 Y\|_{\dot{B}_{2,1}^{\frac{5}{2}}} + \|\nabla Y\|_{\dot{B}_{2,1}^{\frac{5}{2}}} |\partial_3 Y|_0) \lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_3.$$

In general, for $s > 0$, we deduce from (4.1) that

$$\|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^s} \lesssim |\mathcal{A}\mathcal{A}^t - Id|_0 \|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^s} + \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}_{2,1}^s} |\nabla \mathbf{p}_1|_0 + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{2,1}^s},$$

from which and (2.41), we infer

$$\|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^s} \lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}_{2,1}^s} |\nabla \mathbf{p}_1|_0 + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{2,1}^s}.$$

Yet it follows from the product law (5.2) that

$$\begin{aligned} \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{2,1}^s} &\lesssim \|\partial_3 Y \otimes \partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} |\partial_3 Y \otimes \partial_3 Y|_0 \\ &\quad + \|\nabla Y\|_{\dot{B}_{2,1}^s} (|\partial_3 Y|_1 |\partial_3 Y|_0 + |\nabla Y|_1 |\partial_3 Y|_0^2), \end{aligned}$$

which together with (2.41) and (5.8) ensures that

$$(5.9) \quad \|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^s} \lesssim |\partial_3 Y|_0 \left(\|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y\|_{\dot{B}_{2,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|\partial_3 Y\|_3 \right).$$

Exactly along the same line, we have

$$(5.10) \quad \|\nabla \mathbf{p}_2\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \lesssim |Y_t|_0 \|Y_t\|_3 \quad \text{and}$$

$$(5.11) \quad \|\nabla \mathbf{p}_2\|_{\dot{B}_{2,1}^s} \lesssim |Y_t|_0 \left(\|Y_t\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y\|_{\dot{B}_{2,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|Y_t\|_3 \right).$$

• Estimate of $\|\nabla \mathbf{p}'_m(Y; X)\|_0$.

We first deduce from (4.8) that

$$(5.12) \quad \begin{aligned} \|\nabla \mathbf{p}'_1(Y; X)\|_0 &\lesssim \delta_1 \|\nabla \mathbf{p}'_1(Y; X)\|_0 + \|\mathcal{A}(\nabla X \mathcal{A} + \mathcal{A}^t \nabla X) \mathcal{A}^t \nabla \mathbf{p}_1\|_0 \\ &\quad + \|\mathcal{A} \operatorname{div}(\mathcal{A} \nabla X \mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_0 + \|\mathcal{A} \nabla X \mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_0 \\ &\quad + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y))\|_0. \end{aligned}$$

We observe that

$$\|\mathcal{A} \nabla X \mathcal{A} \mathcal{A}^t \nabla \mathbf{p}_1\|_0 \lesssim \|\nabla X\|_{L^6} \|\nabla \mathbf{p}_1\|_{L^3}.$$

Yet it follows by a similar derivation of (5.6) that

$$(5.13) \quad \|\nabla \mathbf{p}_1\|_{L^3} \lesssim \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}} \lesssim |\partial_3 Y|_1 \|\partial_3 Y\|_{L^3} \lesssim |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}},$$

so that

$$\|\mathcal{A} \nabla X \mathcal{A} \mathcal{A}^t \nabla \mathbf{p}_1\|_0 \leq \|\nabla X\|_{L^6} \|\nabla \mathbf{p}_1\|_{L^3} \lesssim |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} \|\nabla X\|_1.$$

Let us handle the remaining terms in (5.12). Indeed with the assumption (2.41), a direct calculation shows that

$$\begin{aligned} \|\mathcal{A} \operatorname{div}(\mathcal{A} \nabla X \mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_0 &\lesssim \|\mathcal{A} \nabla X\|_1 |\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)|_1 \lesssim |\partial_3 Y|_1^2 \|\nabla X\|_1, \\ \|\mathcal{A} \nabla X \mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_0 &\lesssim \|\nabla X\|_0 |\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)|_1 \lesssim |\partial_3 Y|_1^2 \|\nabla X\|_0, \\ \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y))\|_0 &\lesssim \|\partial_3 Y \otimes \partial_3 X\|_1 \lesssim |\partial_3 Y|_1 \|\partial_3 X\|_1. \end{aligned}$$

Substituting the above estimates into (5.12) leads to

$$(5.14) \quad \|\nabla \mathbf{p}'_1(Y; X)\|_0 \lesssim (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) \|\nabla X\|_1 + |\partial_3 Y|_1 \|\partial_3 X\|_1.$$

The same procedure gives rise to

$$(5.15) \quad \|\nabla \mathbf{p}_2\|_{L^3} \lesssim |Y_t|_1^{\frac{4}{3}} \|Y_t\|_0^{\frac{2}{3}}; \quad \text{and}$$

$$(5.16) \quad \|\nabla \mathbf{p}'_2(Y; X)\|_0 \lesssim (|Y_t|_1^{\frac{4}{3}} \|Y_t\|_0^{\frac{2}{3}} + |Y_t|_1^2) \|\nabla X\|_1 + |Y_t|_1 \|X_t\|_1.$$

• Estimate of $\|\nabla \mathbf{p}'_m(Y; X)\|_{\dot{B}_{1,1}^s}$ with $s > 0$

For any $s > 0$, we deduce from (4.8) that

$$\begin{aligned}
 \|\nabla \mathbf{p}'_1(Y; X)\|_{\dot{B}_{1,1}^s} &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}'_1(Y; X)\|_{\dot{B}_{1,1}^s} + \|\mathcal{A}(\nabla X\mathcal{A} + \mathcal{A}^t\nabla X)\mathcal{A}^t\nabla \mathbf{p}_1\|_{\dot{B}_{1,1}^s} \\
 (5.17) \quad &+ \|\mathcal{A}\text{div}(\mathcal{A}\nabla X\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{1,1}^s} + \|\mathcal{A}\nabla X\mathcal{A}\text{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{1,1}^s} \\
 &+ \|\mathcal{A}\text{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y))\|_{\dot{B}_{1,1}^s}.
 \end{aligned}$$

It follows from (5.1) that

$$\|(\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}'_1(Y; X)\|_{\dot{B}_{1,1}^s} \lesssim \delta_1 \|\nabla \mathbf{p}'_1(Y; X)\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla \mathbf{p}'_1(Y; X)\|_0.$$

And applying (5.2) and (5.1) gives

$$\|\mathcal{A}(\nabla X\mathcal{A} + \mathcal{A}^t(\nabla X)^t)\mathcal{A}^t\nabla \mathbf{p}_1\|_{\dot{B}_{1,1}^s} \lesssim \|\nabla \mathbf{p}_1\|_0 \|\nabla X\|_{\dot{B}_{2,1}^s} + (\|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} |\nabla \mathbf{p}_1|_0) \|\nabla X\|_0,$$

and

$$\begin{aligned}
 &\|\mathcal{A}\text{div}(\mathcal{A}\nabla X\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{1,1}^s} \\
 &\lesssim \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_0 + \|\nabla X\|_0 \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{B}_{2,1}^{s+1}} \\
 &\quad + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} \|\nabla X\|_0 |\partial_3 Y|_0^2 + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla X\|_1 |\partial_3 Y|_1 |\partial_3 Y|_0 \\
 &\lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_0 \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + |\partial_3 Y|_0 \left(\|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^s}) |\partial_3 Y|_1 \right) \|\nabla X\|_1.
 \end{aligned}$$

Exactly along the same line, we find

$$\begin{aligned}
 \|\mathcal{A}\nabla X\mathcal{A}\text{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{B}_{1,1}^s} &\lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_{\dot{H}^1} \|\nabla X\|_{\dot{B}_{2,1}^s} \\
 &+ |\partial_3 Y|_0 \left(\|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^s}) |\partial_3 Y|_1 \right) \|\nabla X\|_0,
 \end{aligned}$$

and

$$\begin{aligned}
 \|\mathcal{A}\text{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y))\|_{\dot{B}_{1,1}^s} &\lesssim \|\partial_3 Y\|_0 \|\partial_3 X\|_{\dot{B}_{2,1}^{s+1}} + \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} \|\partial_3 X\|_0 \\
 &+ \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} |\partial_3 Y|_0 \|\partial_3 X\|_0 + \|\nabla Y\|_{\dot{B}_{2,1}^s} |\partial_3 Y|_1 \|\partial_3 X\|_1.
 \end{aligned}$$

Substituting the above estimates into (5.17) and using the estimates (5.6), (5.8), (5.9) and (5.14), we obtain

$$\begin{aligned}
 \|\nabla \mathbf{p}'_1(Y; X)\|_{\dot{B}_{1,1}^s} &\lesssim \mathbf{g}_1(\partial_3 Y, \partial_3 X) \quad \text{with} \\
 (5.18) \quad \mathbf{g}_1(\mathbf{x}, \mathbf{y}) &\stackrel{\text{def}}{=} \|\mathbf{x}\|_0 \|\mathbf{y}\|_{\dot{B}_{2,1}^{s+1}} + |\mathbf{x}|_0 (\|\mathbf{x}\|_0 \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + \|\mathbf{x}\|_1 \|\nabla X\|_{\dot{B}_{2,1}^s}) \\
 &+ (\|\mathbf{x}\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^s}) |\mathbf{x}|_1) \|\mathbf{y}\|_1 \\
 &+ |\mathbf{x}|_1 (\|\mathbf{x}\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^s}) |\mathbf{x}|_3) \|\nabla X\|_1.
 \end{aligned}$$

The same procedure gives rise to

$$(5.19) \quad \|\nabla \mathbf{p}'_2(Y; X)\|_{\dot{B}_{1,1}^s} \lesssim \mathbf{g}_1(Y_t, X_t).$$

Inserting the estimates (5.6), (5.8), (5.9), (5.14) and (5.18) into (5.5) for $m = 1$ yields

$$(5.20) \quad \|f'_1(Y; X)\|_{\dot{B}_{1,1}^s} \lesssim \mathbf{g}_1(\partial_3 Y, \partial_3 X).$$

While by inserting the estimates (5.7), (5.10), (5.11), (5.15), (5.16) and (5.19) into (5.5) for $m = 2$, we obtain

$$(5.21) \quad \|f'_2(Y; X)\|_{\dot{B}_{1,1}^s} \lesssim \mathbf{g}_1(Y_t, X_t).$$

Let us now complete the proof of Proposition 2.3.

Proof of Proposition 2.3. Note that for $s_1 < s < s_2$ and $\alpha = \frac{s_2-s}{s_2-s_1}$, one has

$$\|f\|_{\dot{B}_{2,1}^s} \leq C \left(\frac{1}{s-s_1} + \frac{1}{s_2-s} \right) \|f\|_{\dot{H}^{s_1}}^\alpha \|f\|_{\dot{H}^{s_2}}^{1-\alpha}.$$

In particular, for $s > 0$, this yields

$$(5.22) \quad \|f\|_{\dot{B}_{2,1}^s} \leq C (\|f\|_0 + \|f\|_{\dot{H}^{[s]+1}}) \leq C \|f\|_{[s]+1}.$$

On the other hand, recall (2.39), we deduce from (5.4) that

$$\begin{aligned} \|f'_0(Y; X)\|_{\delta, N} &\lesssim \|\nabla Y\|_0 (\|\nabla X_t\|_{\dot{B}_{2,1}^{2\delta+1}} + \|\nabla X_t\|_{\dot{B}_{2,1}^{N+5}}) \\ &\quad + (\|\nabla Y\|_{\dot{B}_{2,1}^{2\delta+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{N+5}}) \|\nabla X_t\|_0 + \|\nabla Y_t\|_0 (\|\nabla X\|_{\dot{B}_{2,1}^{2\delta+1}} + \|\nabla X\|_{\dot{B}_{2,1}^{N+5}}) \\ &\quad + (\|\nabla Y_t\|_{\dot{B}_{2,1}^{2\delta+1}} + \|\nabla Y_t\|_{\dot{B}_{2,1}^{N+5}} + (\|\nabla Y\|_{\dot{B}_{2,1}^{2\delta+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{N+5}}) \|\nabla Y_t\|_0) \|\nabla X\|_0, \end{aligned}$$

which together with (5.22) ensures (2.42). Along the same line, we deduce (2.43) and (2.44) from (5.20) and (5.21) respectively. This completes the proof of Proposition 2.3. \square

5.2. The estimate of $\| |D|^{-1} f'(Y; X) \|_N$. The purpose of this subsection is to prove Proposition 2.4. We split its proof into the following steps:

5.2.1. *The estimate of $\| |D|^{-1} f'_0(Y; X) \|_N$.* We first deduce from (4.6) that

$$(5.23) \quad \| |D|^{-1} f'_0(Y; X) \|_N \lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\|_N + \|\mathcal{A}(\nabla X\mathcal{A} + \mathcal{A}^t(\nabla X)^t)\mathcal{A}^t\nabla Y_t\|_N.$$

Applying Moser type inequality and using (2.41) gives

$$\begin{aligned} \|(\mathcal{A}\mathcal{A}^t - Id)\nabla X_t\|_N &\lesssim |\nabla Y|_0 \|\nabla X_t\|_N + |\nabla Y|_N \|\nabla X_t\|_0, \\ \|\mathcal{A}\nabla X\mathcal{A}\mathcal{A}^t\nabla Y_t\|_N &\lesssim |\nabla Y_t|_0 \|\nabla X\|_N + (|\nabla Y_t|_N + |\nabla Y_t|_0 |\nabla Y|_N) \|\nabla X\|_0. \end{aligned}$$

Substituting the above estimates into (5.23) leads to (2.45).

5.2.2. *L^2 -estimates for $f'_m(Y; X)$.* We shall divide the proof of (2.46) and (2.47) into the following steps:

(i) Estimates of $\| |D|^{-1} f'_m(Y; X) \|_0$.

By virtue of (4.7), we have

$$(5.24) \quad \| |D|^{-1} f'_m(Y; X) \|_0 \leq \| |D|^{-1} \mathcal{A}^t(\nabla X)^t \mathcal{A}^t(\nabla \mathbf{p}_m)(Y) \|_0 + \| \mathcal{A}^t \nabla \mathbf{p}'_m(Y; X) \|_0.$$

It follows from the law of product in Besov spaces and the imbedding: $L^{\frac{6}{5}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-1}(\mathbb{R}^3)$, that

$$(5.25) \quad \| |D|^{-1} \mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m \|_0 \leq (1 + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \|(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{H}^{-1}} \leq C \|\nabla X\|_0 \|\nabla \mathbf{p}_m\|_{L^3},$$

from which (5.13) and (5.15), we infer

$$\begin{aligned} \| |D|^{-1} \mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_1 \|_0 &\leq C |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} \|\nabla X\|_0, \\ \| |D|^{-1} \mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_2 \|_0 &\leq C |Y|_1^{\frac{4}{3}} \|Y_t\|_0^{\frac{2}{3}} \|\nabla X\|_0. \end{aligned}$$

Similarly, we get, by applying the law of product in Besov spaces, that

$$\| |D|^{-1} \mathcal{A}^t \nabla \mathbf{p}'_m(Y; X) \|_0 \lesssim (1 + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \|\nabla \mathbf{p}'_m(Y; X)\|_{\dot{H}^{-1}}.$$

To deal with the estimate of $\|\nabla \mathbf{p}'_m(Y; X)\|_{\dot{H}^{-1}}$, we deduce from (4.8) and a similar derivation of (5.25) that

$$\begin{aligned} \|\mathbf{p}'_1(Y; X)\|_0 &\lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\nabla \mathbf{p}'_1(Y; X)\|_{\dot{H}^{-1}} + \|\nabla X\|_0 \|\nabla \mathbf{p}_1\|_{L^3} + (1 + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \times \\ &\quad \times \left(\|\mathcal{A}\nabla X \mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_0 + \|\nabla X\|_0 \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{L^3} + \|\mathcal{A}\partial_3 Y \otimes \partial_3 X\|_0 \right) \\ &\lesssim \delta_1 \|\mathbf{p}'_1(Y; X)\|_0 + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0^2) \|\nabla X\|_0 + |\partial_3 Y|_0 \|\partial_3 X\|_0, \end{aligned}$$

which together with (2.41) ensures that

$$(5.26) \quad \|\mathbf{p}'_1(Y; X)\|_0 \lesssim (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0^2) \|\nabla X\|_0 + |\partial_3 Y|_0 \|\partial_3 X\|_0.$$

Exactly along the same line, we deduce from (4.9) that

$$(5.27) \quad \|\mathbf{p}'_2(Y; X)\|_0 \lesssim (|Y_t|_1^{\frac{4}{3}} |Y_t|_0^{\frac{2}{3}} + |Y_t|_0^2) \|\nabla X\|_0 + |Y_t|_0 \|X_t\|_0.$$

Inserting the above estimates into (5.24) leads to

$$(5.28) \quad \| |D|^{-1} f'_1(Y; X) \|_0 \lesssim (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0^2) \|\nabla X\|_0 + |\partial_3 Y|_0 \|\partial_3 X\|_0,$$

$$(5.29) \quad \| |D|^{-1} f'_2(Y; X) \|_0 \lesssim (|Y_t|_1^{\frac{4}{3}} |Y_t|_0^{\frac{2}{3}} + |Y_t|_0^2) \|\nabla X\|_0 + |Y_t|_0 \|X_t\|_0.$$

(ii) Estimates of $\|f'_m(Y; X)\|_{\dot{H}^k}$ for $k \geq 0$ By (4.7) we have

$$(5.30) \quad \|f'_m(Y; X)\|_{\dot{H}^k} \leq \|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{H}^k} + \|\mathcal{A}^t \nabla(\mathbf{p}'_m(Y; X))\|_{\dot{H}^k}.$$

• Estimates for $\|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{H}^k}$.

We get, by applying Moser type inequalities, that

$$\begin{aligned} \|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_m\|_{\dot{H}^k} &\lesssim \|\mathcal{A}^t(\nabla X)^t\|_{L^6} \|D^k(\mathcal{A}^t \nabla \mathbf{p}_m)\|_{L^3} + \|D^k(\mathcal{A}^t(\nabla X)^t)\|_{L^6} \|\mathcal{A}^t \nabla \mathbf{p}_m\|_{L^3} \\ &\lesssim \|D^k \nabla X\|_{L^6} \|\nabla \mathbf{p}_m\|_{L^3} + \|\nabla X\|_{L^6} (\|D^k \nabla \mathbf{p}_m\|_{L^3} + \|D^k \mathcal{A}|_0\| \|\nabla \mathbf{p}_m\|_{L^3}). \end{aligned}$$

Here and in all that follows, we always denote $D^k = \sum_{|\alpha|=k} \partial^\alpha$.

In view of (4.1), applying Moser type inequalities yields

$$\|D^k \nabla \mathbf{p}_1\|_{L^3} \leq \|\mathcal{A}\mathcal{A}^t - Id\|_0 \|D^k \nabla \mathbf{p}_1\|_{L^3} + \|\mathcal{A}\mathcal{A}^t - Id\|_k \|\nabla \mathbf{p}_1\|_{L^3} + \|D^{k+1}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{L^3},$$

from which and (2.41), we infer

$$\|D^k \nabla \mathbf{p}_1\|_{L^3} \lesssim |\nabla Y|_k \|\nabla \mathbf{p}_1\|_{L^3} + \|D^{k+1}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{L^3}.$$

While it is easy to observe that

$$\begin{aligned} \|D^{k+1}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{L^3} &\lesssim \|D^{k+1}(\partial_3 Y \otimes \partial_3 Y)\|_{L^3} + \|D^{k+1} \mathcal{A}|_0\| \|\partial_3 Y \otimes \partial_3 Y\|_{L^3} \\ &\lesssim |\partial_3 Y|_{k+1} \|\partial_3 Y\|_{L^3} + |\nabla Y|_{k+1} |\partial_3 Y|_0 \|\partial_3 Y\|_{L^3}, \end{aligned}$$

which together with (5.13) ensures that

$$\|D^k \nabla \mathbf{p}_1\|_{L^3} \leq (|\partial_3 Y|_{k+1} |\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{k+1} |\partial_3 Y|_0^{\frac{4}{3}}) \|\partial_3 Y\|_0^{\frac{2}{3}},$$

and hence, we obtain

$$(5.31) \quad \begin{aligned} \|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_1\|_{\dot{H}^k} &\leq |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} \|\nabla X\|_{\dot{H}^{k+1}} \\ &\quad + (|\partial_3 Y|_{k+1} |\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{k+1} |\partial_3 Y|_0^{\frac{4}{3}}) \|\partial_3 Y\|_0^{\frac{2}{3}} \|\nabla X\|_{\dot{H}^1}. \end{aligned}$$

By the same procedure, we can show that

$$\|D^k \nabla \mathbf{p}_2\|_{L^3} \leq (|Y_t|_{k+1} |Y_t|_0^{\frac{1}{3}} + |\nabla Y|_{k+1} |Y_t|_0^{\frac{4}{3}}) \|Y_t\|_0^{\frac{2}{3}}$$

and

$$(5.32) \quad \begin{aligned} \|\mathcal{A}^t(\nabla X)^t \mathcal{A}^t \nabla \mathbf{p}_2\|_{\dot{H}^k} &\leq |Y_t|_1^{\frac{4}{3}} |Y_t|_0^{\frac{2}{3}} \|\nabla X\|_{\dot{H}^{k+1}} \\ &\quad + (|Y_t|_{k+1} |Y_t|_0^{\frac{1}{3}} + |\nabla Y|_{k+1} |Y_t|_0^{\frac{4}{3}}) |Y_t|_0^{\frac{2}{3}} \|\nabla X\|_{\dot{H}^1}. \end{aligned}$$

Furthermore, there hold

$$(5.33) \quad \|\nabla \mathbf{p}_1\|_{W^{N,3}} \leq (|\partial_3 Y|_{N+1} |\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{N+1} |\partial_3 Y|_0^{\frac{4}{3}}) \|\partial_3 Y\|_0^{\frac{2}{3}},$$

$$(5.34) \quad \|\nabla \mathbf{p}_2\|_{W^{N,3}} \leq (|Y_t|_{N+1} |Y_t|_0^{\frac{1}{3}} + |\nabla Y|_{N+1} |Y_t|_0^{\frac{4}{3}}) \|Y_t\|_0^{\frac{2}{3}}.$$

• Estimates of $\|\mathcal{A}^t \nabla (\mathbf{p}'_m(Y; X))\|_{\dot{H}^k}$.

Applying Moser type inequality gives

$$(5.35) \quad \|\mathcal{A}^t \nabla (\mathbf{p}'_m(Y; X))\|_{\dot{H}^k} \leq \|\nabla (\mathbf{p}'_m(Y; X))\|_{\dot{H}^k} + |\mathcal{A}^t - Id|_k \|\nabla (\mathbf{p}'_m(Y; X))\|_0.$$

Yet in view of (4.8), we have

$$\begin{aligned} \|\nabla \mathbf{p}'_1(Y; X)\|_{\dot{H}^k} &\lesssim \|(\mathcal{A} \mathcal{A}^t - Id) \nabla \mathbf{p}'_1(Y; X)\|_{\dot{H}^k} + \|\mathcal{A}(\nabla X \mathcal{A} + \mathcal{A}^t \nabla X) \mathcal{A}^t \nabla \mathbf{p}_1\|_{\dot{H}^k} \\ &\quad + \|\mathcal{A} \operatorname{div}(\mathcal{A} \nabla X \mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{H}^k} + \|\mathcal{A} \nabla X \mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{H}^k} \\ &\quad + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y))\|_{\dot{H}^k}. \end{aligned}$$

It follows from a similar derivation of (5.31) that

$$\begin{aligned} \|\mathcal{A} \nabla X \mathcal{A} \mathcal{A}^t \nabla \mathbf{p}_1\|_{\dot{H}^k} &\leq |\partial_3 Y|_1^{\frac{4}{3}} |\partial_3 Y|_0^{\frac{2}{3}} \|\nabla X\|_{\dot{H}^{k+1}} \\ &\quad + (|\partial_3 Y|_{k+1} |\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{k+1} |\partial_3 Y|_0^{\frac{4}{3}}) |\partial_3 Y|_0^{\frac{2}{3}} \|\nabla X\|_{\dot{H}^1} \end{aligned}$$

And we get, by applying Moser type inequality, that

$$\|(\mathcal{A} \mathcal{A}^t - Id) \nabla \mathbf{p}'_1(Y; X)\|_{\dot{H}^k} \leq C \delta_1 \|\nabla \mathbf{p}'_1(Y; X)\|_{\dot{H}^k} + |\nabla Y|_k \|\nabla \mathbf{p}'_1(Y; X)\|_0,$$

and

$$\|\mathcal{A} \operatorname{div}(\mathcal{A} \nabla X \mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{H}^k} \lesssim |\partial_3 Y|_0^2 \|\nabla X\|_{\dot{H}^{k+1}} + (|\partial_3 Y|_{k+1} |\partial_3 Y|_0 + |\nabla Y|_{k+1} |\partial_3 Y|_0^2) \|\nabla X\|_0,$$

and

$$\begin{aligned} \|\mathcal{A} \nabla X \mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_{\dot{H}^k} &\lesssim |\partial_3 Y|_1 |\partial_3 Y|_0 \|\nabla X\|_{\dot{H}^k} \\ &\quad + (|\partial_3 Y|_{k+1} |\partial_3 Y|_0 + |\nabla Y|_{k+1} |\partial_3 Y|_0^2) \|\nabla X\|_0, \end{aligned}$$

and finally

$$\begin{aligned} \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y))\|_{\dot{H}^k} &\lesssim |\partial_3 Y|_0 \|\partial_3 X\|_{\dot{H}^{k+1}} \\ &\quad + (|\partial_3 Y|_{k+1} + |\nabla Y|_{k+1} |\partial_3 Y|_0) \|\partial_3 X\|_0. \end{aligned}$$

As a result, by virtue of (5.14), it comes out

$$(5.36) \quad \begin{aligned} \|\nabla \mathbf{p}'_1(Y; X)\|_{\dot{H}^k} &\lesssim \mathfrak{g}_2(\partial_3 Y, \partial_3 X) \quad \text{with} \\ \mathfrak{g}_2(\mathfrak{x}, \mathfrak{y}) &\stackrel{\text{def}}{=} (|\mathfrak{x}|_1^{\frac{4}{3}} \|\mathfrak{x}\|_0^{\frac{2}{3}} + |\mathfrak{x}|_1^2) (\|\nabla X\|_{\dot{H}^{k+1}} + |\nabla Y|_{k+1} \|\nabla X\|_1) + |\mathfrak{x}|_0 \|\mathfrak{y}\|_{\dot{H}^{k+1}} \\ &\quad + (|\mathfrak{x}|_{k+1} + |\nabla Y|_{k+1} |\mathfrak{x}|_1) \|\mathfrak{y}\|_1 + |\mathfrak{x}|_{k+1} (|\mathfrak{x}|_0^{\frac{1}{3}} \|\mathfrak{x}\|_0^{\frac{2}{3}} + |\mathfrak{x}|_0) \|\nabla X\|_1. \end{aligned}$$

Substituting the above estimate and (5.14) into (5.35) for $m = 1$ shows that $\|\mathcal{A}^t \nabla (\mathbf{p}'_1(Y; X))\|_{\dot{H}^k}$ shares the same estimate as above.

Similarly, we can show that

$$(5.37) \quad \|\nabla \mathbf{p}'_2(Y; X)\|_{\dot{H}^k} \lesssim \mathfrak{g}_2(Y_t, X_t).$$

Substituting the above estimate and (5.16) into (5.35) for $m = 2$ shows that $\|\mathcal{A}^t \nabla (\mathbf{p}'_2(Y; X))\|_{\dot{H}^k}$ shares the same estimate as above.

Let us now turn to the estimates of $\|f'_1(Y; X)\|_{\dot{H}^k}$ and $\|f'_2(Y; X)\|_{\dot{H}^k}$. As a matter of fact, by inserting (5.31) and (5.36) into (5.30) for $m = 1$, we achieve

$$(5.38) \quad \|f'_1(Y; X)\|_{\dot{H}^k} \lesssim \mathfrak{g}_2(\partial_3 Y, \partial_3 X).$$

Similarly by inserting (5.32) and (5.37) into (5.30) for $m = 2$, we obtain

$$(5.39) \quad \|f'_2(Y; X)\|_{\dot{H}^k} \lesssim \mathfrak{g}_2(Y_t, X_t).$$

Now we are in a position to complete the proof of Proposition 2.4.

Proof of Proposition 2.4. It remains to prove (2.46) and (2.47). Indeed, combining (5.28) with (5.38), we obtain (2.46). While combining (5.29) with (5.39) leads to (2.47). This completes the proof of Proposition 2.4. \square

6. ENERGY ESTIMATES FOR THE LINEARIZED EQUATION

The goal of this section is to present the proof of Theorem 2.3.

6.1. First-order energy estimates. Let us first carry out the estimate of $\mathcal{E}_0(t)$ (2.50).

• The estimate of $\|\nabla X\|_0$.

We first get, by taking L^2 inner product of (2.40) with X , that

$$(6.1) \quad \frac{d}{dt} \left(\frac{1}{2} \|\nabla X\|_0^2 + (X_t | X)_{L^2} \right) + \|\partial_3 X\|_0^2 - \|X_t\|_0^2 = (f'(Y; X) + g | X)_{L^2}.$$

And it follows by taking L^2 inner product of (2.40) with $(-\Delta)^{-1} X_t$ that

$$\frac{1}{2} \frac{d}{dt} (\| |D|^{-1} X_t \|_0^2 + \| |D|^{-1} \partial_3 X \|_0^2) + \|X_t\|_0^2 = ((-\Delta)^{-1} (f'(Y; X) + g | X))_{L^2}.$$

Summing up the above equality with $\frac{1}{4} \times (6.1)$ yields

$$(6.2) \quad \begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} (\| |D|^{-1} X_t \|_0^2 + \| |D|^{-1} \partial_3 X \|_0^2 + \frac{1}{4} \|\nabla X\|_0^2) + \frac{1}{4} (X_t | X)_{L^2} \right) \\ & + \frac{3}{4} \|X_t\|_0^2 + \frac{1}{4} \|\partial_3 X\|_0^2 = (|D|^{-1} (f'(Y; X) + g | X) | \frac{1}{4} |D| X + |D|^{-1} X_t)_{L^2}. \end{aligned}$$

It is easy to observe that

$$\begin{aligned} & |(|D|^{-1} \nabla \cdot (\mathcal{A}(\nabla X \mathcal{A} + \mathcal{A}^t (\nabla X)^t) \mathcal{A}^t) \nabla Y_t) | \frac{1}{4} |D| X + |D|^{-1} X_t)_{L^2} | \\ & \leq C |\nabla Y_t|_0 \|\nabla X\|_0 (\|\nabla X\|_0 + \| |D|^{-1} X_t \|_0), \end{aligned}$$

and

$$\begin{aligned} & (|D|^{-1} \nabla \cdot ((\mathcal{A} \mathcal{A}^t - Id) \nabla X_t) | |D| X)_{L^2} = -((\mathcal{A} \mathcal{A}^t - Id) \nabla X_t | \nabla X)_{L^2} \\ & = -\frac{1}{2} \frac{d}{dt} ((\mathcal{A} \mathcal{A}^t - Id) \nabla X | \nabla X)_{L^2} + \int_{\mathbb{R}^3} \partial_t (\mathcal{A} \mathcal{A}^t) |\nabla X|^2 dx, \end{aligned}$$

and

$$|(|D|^{-1} \nabla \cdot ((\mathcal{A} \mathcal{A}^t - Id) \nabla X_t) | |D|^{-1} X_t)_{L^2} | \lesssim \| \mathcal{A} \mathcal{A}^t - Id \|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\nabla X_t\|_{\dot{H}^{-1}} \|X_t\|_0 \leq C \delta_1 \|X_t\|_0^2.$$

Hence in view of (4.6), under the assumption of (2.41), by taking δ_1 so small that $C \delta_1 \leq \frac{1}{4}$, we obtain

$$(6.3) \quad \begin{aligned} & |(|D|^{-1} f'_0(Y; X) | \frac{1}{4} |D| X + |D|^{-1} X_t)_{L^2} + \frac{1}{8} \frac{d}{dt} ((\mathcal{A} \mathcal{A}^t - Id) \nabla X | \nabla X)_{L^2} | \\ & \leq C |\nabla Y_t|_0 \|\nabla X\|_0 (\|\nabla X\|_0 + \| |D|^{-1} X_t \|_0) + \frac{1}{4} \|X_t\|_0^2. \end{aligned}$$

While by virtue of (5.28) and (5.29), we have

$$\begin{aligned}
 & |(|D|^{-1}(f'_1(Y; X) + f'_2(Y; X))| \frac{1}{4}|D|X + |D|^{-1}X_t)_{L^2}| \\
 (6.4) \quad & \leq \frac{1}{8}(\|X_t\|_0^2 + \|\partial_3 X\|_0^2) + C\left(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0^2 \right. \\
 & \quad \left. + |Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}} + |Y_t|_0^2\right)(\|\nabla X\|_0^2 + \||D|^{-1}X_t\|_0^2).
 \end{aligned}$$

Inserting (6.3) and (6.4) into (6.2) gives rise to

$$\begin{aligned}
 & \frac{d}{dt} \left(\frac{1}{2}(\||D|^{-1}X_t\|_0^2 + \||D|^{-1}\partial_3 X\|_0^2 + \frac{1}{4}(\mathcal{A}\mathcal{A}^t\nabla X|\nabla X)_{L^2}) + \frac{1}{4}(X_t|X)_{L^2} \right) \\
 (6.5) \quad & + \frac{1}{8}(\|X_t\|_0^2 + \|\partial_3 X\|_0^2) \leq \||D|^{-1}g\|_0(\|\nabla X\|_0 + \||D|^{-1}X_t\|_0) \\
 & + C\left(|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0^2 + |Y_t|_1\right)(\|\nabla X\|_0^2 + \||D|^{-1}X_t\|_0^2),
 \end{aligned}$$

by applying the assumption (2.49).

On the other hand, since $\mathcal{A}\mathcal{A}^t$ is a positive definite matrix ($|\mathcal{A}\mathcal{A}^t - Id|_0 \leq C\delta_1 \leq \frac{1}{4}$), it holds that

$$(\mathcal{A}\mathcal{A}^t\nabla X|\nabla X)_{L^2} \geq (1 - C\delta_1)\|\nabla X\|_0^2 \geq \frac{3}{4}\|\nabla X\|_0^2,$$

so that one has

$$\begin{aligned}
 (6.6) \quad & \frac{1}{2}(\||D|^{-1}X_t\|_0^2 + \||D|^{-1}\partial_3 X\|_0^2 + \frac{1}{4}(\mathcal{A}\mathcal{A}^t\nabla X|\nabla X)_{L^2}) + \frac{1}{4}(X_t|X)_{L^2} \\
 & \geq \frac{1}{4}\||D|^{-1}X_t\|_0^2 + \frac{1}{2}\||D|^{-1}\partial_3 X\|_0^2 + \frac{1}{32}\|\nabla X\|_0^2.
 \end{aligned}$$

•The estimate of $\|X_t\|_0$.

Multiplying (2.40) by X_t and integrating the resulting equality over \mathbb{R}^3 , we get

$$\frac{1}{2}\frac{d}{dt}(\|X_t\|_0^2 + \|\partial_3 X\|_0^2) + \|\nabla X_t\|_0^2 = (f'(Y; X) + g|X_t)_{L^2}.$$

In view of (4.6), we infer

$$|(f'_0(Y; X)|X_t)_{L^2}| \leq C|\nabla Y_t|_0^2\|\nabla X\|_0^2 + \frac{1}{4}\|\nabla X_t\|_0^2.$$

While it follows from (5.28) to (5.29) that

$$\begin{aligned}
 |(f'_1(Y; X) + f'_2(Y; X)|X_t)_{L^2}| & \leq C\left(|\partial_3 Y|_0\|\partial_3 X\|_0 + |Y_t|_0\|X_t\|_0\right) + (|\partial_3 Y|_1^{\frac{4}{3}}\|\partial_3 Y\|_0^{\frac{2}{3}} \\
 & + |\partial_3 Y|_1^2 + |Y_t|_1^{\frac{4}{3}}\|Y_t\|_0^{\frac{2}{3}} + |Y_t|_1^2)\|\nabla X\|_0\|\nabla X_t\|_0.
 \end{aligned}$$

As a result, thanks to the assumption (2.49), it comes out

$$\begin{aligned}
 (6.7) \quad & \frac{d}{dt}(\|X_t\|_0^2 + \|\partial_3 X\|_0^2) + \|\nabla X_t\|_0^2 \leq C(|\partial_3 Y|_1^{\frac{8}{3}}\|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^4 + |Y_t|_1^2)\|\nabla X\|_0^2 \\
 & + C(|\partial_3 Y|_0^2\|\partial_3 X\|_0^2 + |Y_t|_0^2\|X_t\|_0^2) + 4\||D|^{-1}g\|_0^2.
 \end{aligned}$$

•The estimate of $\|\nabla X_t\|_0$.

By taking L^2 inner product of (2.40) with $-\Delta X_t$ gives

$$(6.8) \quad \frac{1}{2}\frac{d}{dt}(\|\nabla X_t\|_{L^2}^2 + \|\nabla\partial_3 X\|_0^2) + \|\Delta X_t\|_0^2 = -(f'(Y; X) + g|\Delta X_t)_{L^2}.$$

It is easy to observe from (2.41) and (4.6) that

$$(6.9) \quad \|f'_0(Y; X)\|_0 \leq \frac{1}{4} \|\Delta X_t\|_0 + |\nabla Y_t|_1 \|\nabla X\|_1.$$

Then by substituting the estimates (6.9), (5.38) and (5.39) into (6.8) and using the assumptions (2.41) and (2.49), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla X_t\|_{L^2}^2 + \|\nabla \partial_3 X\|_0^2) + \|\Delta X_t\|_0^2 \leq & C \left((|Y_t|_2 + |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) \|\nabla X\|_1 \right. \\ & \left. + |\partial_3 Y|_1 \|\partial_3 X\|_1 + |Y_t|_1 \|X_t\|_1 + \|g\|_0 \right) \|\Delta X_t\|_0, \end{aligned}$$

which implies

$$(6.10) \quad \begin{aligned} \frac{d}{dt} (\|\nabla X_t\|_{L^2}^2 + \|\nabla \partial_3 X\|_0^2) + \|\Delta X_t\|_0^2 \leq & C (|Y_t|_2^2 + |\partial_3 Y|_1^{\frac{8}{3}} \|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^4) \|\nabla X\|_1^2 \\ & + C (|\partial_3 Y|_1^2 \|\partial_3 X\|_1^2 + |Y_t|_1^2 \|X_t\|_1^2) + \|g\|_0^2. \end{aligned}$$

• The estimate of $\|\nabla X\|_{\dot{H}^1}$.

In this step, we shall use the equivalent formulation, (2.53), of (2.40). We first get, by taking L^2 inner product of (2.53) with $-\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)$, that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_0^2 + (\partial_3^2 X | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2} - (X_{tt} | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2} \\ = - (\tilde{f}'(Y; X) + g | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2}. \end{aligned}$$

By using integration by parts, one has

$$\begin{aligned} (X_{tt} | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2} &= - \frac{d}{dt} (\nabla X_t | \mathcal{A}\mathcal{A}^t \nabla X)_{L^2} + (\nabla X_t | \partial_t (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2}, \\ (\partial_3^2 X | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2} &= (\nabla \partial_3 X | \mathcal{A}\mathcal{A}^t \nabla \partial_3 X)_{L^2} + (\nabla \partial_3 X | \partial_3 (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2}. \end{aligned}$$

Since $\mathcal{A}\mathcal{A}^t$ is a positive definite matrix, we infer

$$(6.11) \quad \begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_0^2 + (\nabla X_t | \mathcal{A}\mathcal{A}^t \nabla X)_{L^2} \right) + \frac{1}{2} \|\nabla \partial_3 X\|_0^2 \\ \leq 2 \|\nabla X_t\|_0^2 + \frac{1}{4} \|\nabla \partial_3 X\|_0^2 + C (|\nabla Y_t|_0^2 + |\partial_3 \nabla Y|_0^2) \|\nabla X\|_0^2 \\ - (\tilde{f}'(Y; X) + g | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2}. \end{aligned}$$

Yet under the assumption of (2.41), it is easy to observe from (2.54) that

$$\|\tilde{f}'_0(Y; X)\|_0 \leq C |\nabla Y_t|_1 \|\nabla X\|_1.$$

Whereas it follows from (5.38), (5.39) that

$$\begin{aligned} |(f'_1(Y; X) + f'_2(Y; X) | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{L^2}| \lesssim & \left(|\partial_3 Y|_1 \|\partial_3 X\|_1 + |Y_t|_1 \|X_t\|_1 \right. \\ & \left. + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2 + |Y_t|_1^{\frac{4}{3}} \|Y_t\|_0^{\frac{2}{3}} + |Y_t|_1^2) \|\nabla X\|_1 \right) \|\nabla X\|_1. \end{aligned}$$

Inserting the above estimates into (6.11) yields

$$(6.12) \quad \begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_0^2 + (\nabla X_t | \mathcal{A}\mathcal{A}^t \nabla X)_{L^2} \right) + \frac{1}{8} \|\nabla \partial_3 X\|_0^2 \leq & 3 \|X_t\|_1^2 + \frac{1}{20} \|\partial_3 X\|_0^2 \\ & + \|g\|_0 \|\nabla X\|_1 + C (|\partial_3 Y|_1^2 + |Y_t|_1^2 + |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |Y_t|_1^{\frac{4}{3}} \|Y_t\|_0^{\frac{2}{3}}) \|\nabla X\|_1^2. \end{aligned}$$

Let us denote

$$(6.13) \quad E_0(t) \stackrel{\text{def}}{=} \frac{1}{2} \left(\| |D|^{-1} X_t \|_{H^2}^2 + \| |D|^{-1} \partial_3 X \|_2^2 + \frac{1}{4} (\mathcal{A} \mathcal{A}^t \nabla X | \nabla X)_{L^2} \right) \\ + \frac{1}{4} (X_t | X)_{L^2} + \frac{1}{48} \left(\frac{1}{2} \| \nabla \cdot (\mathcal{A} \mathcal{A}^t \nabla X) \|_0^2 + (\nabla X_t | \mathcal{A} \mathcal{A}^t \nabla X)_{L^2} \right).$$

Then by summing up the inequalities (6.5), (6.7), (6.10) and $\frac{1}{48} \times (6.12)$, we obtain,

$$(6.14) \quad \frac{d}{dt} E_0(t) + \frac{1}{16} \| X_t \|_2^2 + \frac{1}{384} \| \partial_3 X \|_1^2 \leq \| |D|^{-1} g \|_1^2 + C (| \partial_3 Y |_1^{\frac{4}{3}} \| \partial_3 Y \|_0^{\frac{2}{3}} + | \partial_3 Y |_1^2 + | Y_t |_2) \times \\ \times (\| \nabla X \|_1^2 + \| |D|^{-1} X_t \|_0^2 + \| \partial_3 X \|_1^2 + \| X_t \|_1^2) + \| |D|^{-1} g \|_1 (\| \nabla X \|_1 + \| |D|^{-1} X_t \|_0).$$

Notice that

$$(\nabla X_t | \mathcal{A} \mathcal{A}^t \nabla X)_{L^2} \geq - \| X_t \|_0^2 - \frac{1}{4} \| \nabla \cdot (\mathcal{A} \mathcal{A}^t \nabla X) \|_0^2,$$

and

$$\| \nabla \cdot (\mathcal{A} \mathcal{A}^t \nabla X) \|_0 \geq \| \nabla X \|_{\dot{H}^1} - \| (\mathcal{A} \mathcal{A}^t - Id) \nabla X \|_{\dot{H}^1} \geq (1 - C \delta_1) \| \nabla X \|_{\dot{H}^1},$$

so that we deduce from (6.6) and (6.13) that

$$(6.15) \quad E_0(t) \geq \frac{1}{16^2} \left(\| |D|^{-1} X_t \|_2^2 + \| |D|^{-1} \partial_3 X \|_2^2 + \| \nabla X \|_1^2 \right).$$

Hence for any $\varepsilon > 0$, we deduce from (6.14) that

$$(6.16) \quad \frac{d}{dt} E_0(t) + \frac{1}{16} \| X_t \|_2^2 + \frac{1}{384} \| \partial_3 X \|_1^2 \leq \langle t \rangle^{1+\varepsilon} \| |D|^{-1} g \|_1^2 \\ + C_\varepsilon \left(| \partial_3 Y |_1^{\frac{4}{3}} \| \partial_3 Y \|_0^{\frac{2}{3}} + | \partial_3 Y |_1^2 + | Y_t |_2 + \langle t \rangle^{-(1+\varepsilon)} \right) E_0(t).$$

Applying Gronwall's inequality yields for any $\varepsilon > 0$ that

$$E_0(t) + \frac{1}{16} \| X_t \|_{L_t^2(H^2)}^2 + \frac{1}{384} \| \partial_3 X \|_{L_t^2(H^1)}^2 \leq C_\varepsilon \left(\int_0^t \langle s \rangle^{1+\varepsilon} \| |D|^{-1} g(s) \|_1^2 ds \right) \times \\ \times \exp C \left(| \partial_3 Y |_{\frac{1}{2}+\varepsilon,1}^{\frac{4}{3}} \| \partial_3 Y \|_{L_t^2(L^2)}^{\frac{2}{3}} + | \partial_3 Y |_{\frac{1}{2}+\varepsilon,1}^2 + | Y_t |_{1+\varepsilon,2} \right),$$

which together with (6.15) ensures the first inequality of (2.50).

6.2. Higher-order energy estimates. In this subsection, we shall derive the estimates for

$$(6.17) \quad \dot{E}_{k+1}(t) \stackrel{\text{def}}{=} \| \partial_3 X \|_{\dot{H}^{k+1}}^2 + \| X_t \|_{\dot{H}^{k+1}}^2 + \| \nabla X \|_{\dot{H}^{k+1}}^2 \quad \text{for } k \geq 0.$$

We first get, by taking the \dot{H}^{k+1} -inner product of (2.40) with X_t , that

$$\frac{1}{2} \frac{d}{dt} (\| X_t \|_{\dot{H}^{k+1}}^2 + \| \partial_3 X \|_{\dot{H}^{k+1}}^2) + \| X_t \|_{\dot{H}^{k+2}}^2 = (f'(Y; X) + g | X_t)_{\dot{H}^{k+1}},$$

which implies

$$(6.18) \quad \frac{d}{dt} (\| X_t \|_{\dot{H}^{k+1}}^2 + \| \partial_3 X \|_{\dot{H}^{k+1}}^2) + \| X_t \|_{\dot{H}^{k+2}}^2 \leq \| f'(Y; X) \|_{\dot{H}^k}^2 + \| g \|_{\dot{H}^k}^2.$$

Yet in view of (4.6), it follows from Moser type inequality that

$$(6.19) \quad \| f'_0(Y; X) \|_{\dot{H}^k} \lesssim \| \nabla Y_t |_0 \| \| \nabla X \|_{\dot{H}^{k+1}} + (\| \nabla Y_t |_{k+1} + \| \nabla Y_t |_0 \| \| \nabla Y |_{k+1} \|) \| \nabla X \|_0 \\ + \| \nabla Y |_0 \| \| \nabla X_t \|_{\dot{H}^{k+1}} + \| \nabla Y |_{k+1} \| \| \nabla X_t \|_0,$$

from which, (5.38), (5.39) and the assumption (2.49), we infer

$$\begin{aligned}
\|f'(Y; X)\|_{\dot{H}^k} &\lesssim |\partial_3 Y|_0 \|\partial_3 X\|_{\dot{H}^{k+1}} + |Y_t|_0 \|X_t\|_{\dot{H}^{k+1}} + |\nabla Y|_0 \|X_t\|_{\dot{H}^{k+2}} \\
&+ (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2 + |Y_t|_1) (\|\nabla X\|_{\dot{H}^{k+1}} + |\nabla Y|_{k+1} \|\nabla X\|_1) \\
&+ (|\partial_3 Y|_{k+1} + |\nabla Y|_{k+1} |\partial_3 Y|_1) \|\partial_3 X\|_1 + (|Y_t|_{k+1} + |\nabla Y|_{k+1} |Y_t|_1) \|X_t\|_1 \\
&+ |\nabla Y|_{k+1} \|\nabla X_t\|_0 + (|\partial_3 Y|_{k+1} (|\partial_3 Y|_0^{\frac{1}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1) + |Y_t|_{k+2}) \|\nabla X\|_1.
\end{aligned} \tag{6.20}$$

Inserting (6.20) into (6.18), and using the assumption (2.41) so that $|\nabla Y|_0 \leq \delta_1$, we deduce that

$$\begin{aligned}
\frac{d}{dt} (\|X_t\|_{\dot{H}^{k+1}}^2 + \|\partial_3 X\|_{\dot{H}^{k+1}}^2) + \frac{3}{4} \|X_t\|_{\dot{H}^{k+2}}^2 &\lesssim |\partial_3 Y|_0^2 \|\partial_3 X\|_{\dot{H}^{k+1}}^2 + |Y_t|_0^2 \|X_t\|_{\dot{H}^{k+1}}^2 \\
&+ \|g\|_{\dot{H}^k}^2 + (|\partial_3 Y|_1^{\frac{8}{3}} \|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^4 + |Y_t|_1^2) (\|\nabla X\|_{\dot{H}^{k+1}}^2 + |\nabla Y|_{k+1}^2 \|\nabla X\|_1^2) \\
&+ (|\partial_3 Y|_{k+1}^2 + |\nabla Y|_{k+1}^2 |\partial_3 Y|_1^2) \|\partial_3 Y\|_1^2 + (|Y_t|_{k+1}^2 + |\nabla Y|_{k+1}^2 |Y_t|_1^2) \|X_t\|_1^2 \\
&+ |\nabla Y|_{k+1}^2 \|\nabla X_t\|_0^2 + (|\partial_3 Y|_{k+1}^2 (|\partial_3 Y|_0^{\frac{2}{3}} \|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^2) + |Y_t|_{k+2}^2) \|\nabla X\|_1^2.
\end{aligned} \tag{6.21}$$

Secondly, by taking the \dot{H}^k -inner product of (2.53) with $-\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)$, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_{\dot{H}^k}^2 + (\partial_3^2 X | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} \\
- (X_{tt} | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} = -(\tilde{f}'(Y; X) + g | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k}.
\end{aligned} \tag{6.22}$$

By using integration by parts, one has

$$-(X_{tt} | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} = -\frac{d}{dt} (X_t | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} - (\nabla X_t | \partial_t (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k},$$

and

$$\begin{aligned}
|(\nabla X_t | \partial_t (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k}| &\leq \|X_t\|_{\dot{H}^{k+1}} \left(\frac{3}{2} \|\nabla X_t\|_{\dot{H}^k} + |\nabla Y|_k \|\nabla X_t\|_0 \right. \\
&\quad \left. + |\nabla Y_t|_0 \|\nabla X\|_{\dot{H}^k} + |\nabla Y_t|_k \|\nabla X\|_0 \right),
\end{aligned}$$

so that it comes out

$$\begin{aligned}
|(X_{tt} | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} - \frac{d}{dt} (X_t | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k}| \\
\leq 2 \|X_t\|_{\dot{H}^{k+1}}^2 + C_k (|\nabla Y_t|_0^2 \|\nabla X\|_{\dot{H}^k}^2 + |\nabla Y|_k^2 \|\nabla X_t\|_0^2 + |\nabla Y_t|_k^2 \|\nabla X\|_0^2).
\end{aligned}$$

Similarly, again by using integration by parts, one has

$$(\partial_3^2 X | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} = (\nabla \partial_3 X | \mathcal{A}\mathcal{A}^t \nabla \partial_3 X)_{\dot{H}^k} + (\nabla \partial_3 X | \partial_3 (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k}.$$

Since $|\mathcal{A}\mathcal{A}^t - Id|_0 \leq C\delta_1 \leq \frac{1}{4}$ due to (2.41), applying Moser type inequality gives

$$(\nabla \partial_3 X | \mathcal{A}\mathcal{A}^t \nabla \partial_3 X)_{\dot{H}^k} \geq \frac{1}{2} \|\nabla \partial_3 X\|_{\dot{H}^k}^2 - C_k |\nabla Y|_k^2 \|\nabla \partial_3 X\|_0^2,$$

and

$$|(\nabla \partial_3 X | \partial_3 (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k}| \leq \frac{1}{4} \|\nabla \partial_3 X\|_{\dot{H}^k}^2 + C_k (|\partial_3 Y|_1^2 \|\nabla X\|_{\dot{H}^k}^2 + |\partial_3 Y|_{k+1}^2 \|\nabla X\|_0^2),$$

so that there holds

$$(\nabla \partial_3 X | \mathcal{A}\mathcal{A}^t \nabla \partial_3 X)_{\dot{H}^k} \geq \frac{1}{4} \|\nabla \partial_3 X\|_{\dot{H}^k}^2 - C_k (|\nabla Y|_k^2 \|\nabla \partial_3 X\|_0^2 + |\partial_3 Y|_1^2 \|\nabla X\|_{\dot{H}^k}^2 + |\partial_3 Y|_{k+1}^2 \|\nabla X\|_0^2).$$

Inserting the above estimates into (6.22) gives rise to

$$\begin{aligned}
(6.23) \quad & \frac{d}{dt} \left(\frac{1}{2} \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_{\dot{H}^k}^2 - (X_t | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} \right) + \frac{1}{4} \|\partial_3 X\|_{\dot{H}^{k+1}}^2 \\
& \leq 2 \|X_t\|_{\dot{H}^{k+1}}^2 + (|\partial_3 Y|_0^2 + |Y_t|_0^2) \|\nabla X\|_{\dot{H}^{k+1}}^2 + C_k |\nabla Y|_k^2 (\|\nabla \partial_3 X\|_0^2 + \|\nabla X_t\|_0^2) \\
& \quad + C_k (|\partial_3 Y|_{k+1}^2 + |Y_t|_{k+2}^2) \|\nabla X\|_0^2 + (\|\tilde{f}'(Y; X)\|_{\dot{H}^k} + \|g\|_{\dot{H}^k}) \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_{\dot{H}^k}.
\end{aligned}$$

We remark that

$$\begin{aligned}
(6.24) \quad & \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X) - \Delta X\|_{\dot{H}^k} \leq |\mathcal{A}\mathcal{A}^t - Id|_0 \|\nabla X\|_{\dot{H}^{k+1}} + C_k |\mathcal{A}\mathcal{A}^t - Id|_{k+1} \|\nabla X\|_0 \\
& \leq \frac{1}{2} \|\nabla X\|_{\dot{H}^{k+1}} + C_k |\nabla Y|_{k+1} \|\nabla X\|_0.
\end{aligned}$$

Moreover, in view of (2.54), we have

$$(6.25) \quad \|\tilde{f}'_0(Y; X)\|_{\dot{H}^k} \lesssim |Y_t|_1 \|\nabla X\|_{\dot{H}^{k+1}} + (|Y_t|_{k+2} + |Y_t|_1 |\nabla Y|_{k+1}) \|\nabla X\|_0,$$

which together with (5.38) and (5.39) ensures that

$$\begin{aligned}
(6.26) \quad & \|\tilde{f}'(Y; X)\|_{\dot{H}^k} \lesssim |\partial_3 Y|_0 \|\partial_3 X\|_{\dot{H}^{k+1}} + |Y_t|_0 \|X_t\|_{\dot{H}^{k+1}} + (|\partial_3 Y|_{k+1} + |\nabla Y|_{k+1} |\partial_3 Y|_1) \|\partial_3 X\|_1 \\
& + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2 + |Y_t|_1) (\|\nabla X\|_{\dot{H}^{k+1}} + |\nabla Y|_{k+1} \|\nabla X\|_1) + \|X_t\|_1 (|Y_t|_{k+1} \\
& + |\nabla Y|_{k+1} |Y_t|_1) + (|\partial_3 Y|_{k+1} (|\partial_3 Y|_0^{\frac{1}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1) + |Y_t|_{k+2}) \|\nabla X\|_1.
\end{aligned}$$

Inserting the above inequalities to (6.23) yields

$$\begin{aligned}
(6.27) \quad & \frac{d}{dt} \left(\frac{1}{2} \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_{\dot{H}^k}^2 - (X_t | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k} \right) + \frac{1}{8} \|\partial_3 X\|_{\dot{H}^{k+1}}^2 \\
& \leq 3 \|X_t\|_{\dot{H}^{k+1}}^2 + \langle t \rangle^{1+\varepsilon} \|g\|_{\dot{H}^k}^2 + C_k (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2 + |Y_t|_1 + \langle t \rangle^{-(1+\varepsilon)}) \|\nabla X\|_{\dot{H}^{k+1}}^2 \\
& + C_k ((|\partial_3 Y|_{k+1}^2 + |\nabla Y|_{k+1}^2 |\partial_3 Y|_1^2) \langle t \rangle^{1+\varepsilon} + |\nabla Y|_k^2 \|\partial_3 X\|_1^2 \\
& + C_k ((|Y_t|_{k+1}^2 + |\nabla Y|_{k+1}^2 |Y_t|_1^2) \langle t \rangle^{1+\varepsilon} + |\nabla Y|_k^2 \|X_t\|_1^2 \\
& + C_k \left\{ (|\partial_3 Y|_{k+1}^2 (|\partial_3 Y|_0^{\frac{2}{3}} \|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^2) + |Y_t|_{k+2}^2) \langle t \rangle^{1+\varepsilon} + |\partial_3 Y|_{k+1}^2 \right. \\
& \left. + |\nabla Y|_{k+1}^2 ((|\partial_3 Y|_1^{\frac{8}{3}} \|\partial_3 Y\|_0^{\frac{4}{3}} + |\partial_3 Y|_1^4 + |Y_t|_1^2) \langle t \rangle^{1+\varepsilon} + \langle t \rangle^{-(1+\varepsilon)}) \right\} \|\nabla X\|_1^2
\end{aligned}$$

Let us introduce

$$(6.28) \quad \dot{D}_{k+1}(t) \stackrel{\text{def}}{=} \|X_t\|_{\dot{H}^{k+1}}^2 + \|\partial_3 X\|_{\dot{H}^{k+1}}^2 + \frac{1}{2} \|\nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X)\|_{\dot{H}^k}^2 - (X_t | \nabla \cdot (\mathcal{A}\mathcal{A}^t \nabla X))_{\dot{H}^k}.$$

Then it follows from (6.24) that

$$(6.29) \quad \dot{D}_{k+1}(t) \geq \frac{1}{8} \dot{E}_{k+1}(t) - C_{k+1} \|X_t\|_0^2 - C_{k+1} |\nabla Y|_{k+1}^2 \|\nabla X\|_0^2,$$

with $\dot{E}_{k+1}(t)$ being given by (6.17).

Hence by summing up (6.21) and (6.27), and then integrating the resulting inequality over $[0, t]$ and using (6.29), we achieve

$$\begin{aligned}
(6.30) \quad & \dot{E}_{k+1}(t) + \int_0^t \left(\frac{1}{2} \|X_t\|_{\dot{H}^{k+2}}^2 + \frac{1}{8} \|\partial_3 X\|_{\dot{H}^{k+1}}^2 \right) ds \\
& \leq 8\dot{D}_{k+1}(t) + \|X_t\|_{0,0}^2 + |\nabla Y|_{0,k+1}^2 \|\nabla X\|_{0,0}^2 + \int_0^t \left(\frac{1}{2} \|X_t\|_{\dot{H}^{k+2}}^2 + \frac{1}{8} \|\partial_3 X\|_{\dot{H}^{k+1}}^2 \right) ds \\
& \lesssim \int_0^t \left(|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0^2 + |Y_t|_1 + \langle s \rangle^{-(1+\varepsilon)} \right) \dot{E}_{k+1}(s) ds \\
& \quad + \|\langle t \rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(\dot{H}^k)}^2 + \gamma_{\varepsilon,k+1}(Y)^2 \mathcal{E}_0^2(t),
\end{aligned}$$

where $\mathcal{E}_0(t)$ is given by (2.50) and $\gamma_{\varepsilon,k+1}(Y)$ by (2.52). Applying Gronwall's inequality to (6.30) and using (2.50), we obtain

$$\dot{E}_{k+1}(t) \leq C_{\varepsilon,k} \left(\|\langle t \rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(\dot{H}^k)}^2 + \gamma_{\varepsilon,k+1}(Y)^2 \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L_t^2(H^1)}^2 \right) E_{\varepsilon}(Y),$$

from which and (6.30), we infer

$$\begin{aligned}
(6.31) \quad & \|(X_t, \partial_3 X, \nabla X)\|_{L_t^\infty(\dot{H}^{k+1})} + \|X_t\|_{L_t^2(\dot{H}^{k+2})} + \|\partial_3 X\|_{L_t^2(\dot{H}^{k+1})} \\
& \leq C_{\varepsilon,k} \left(\|\langle t \rangle^{\frac{1+\varepsilon}{2}} g\|_{L_t^2(\dot{H}^k)} + \gamma_{\varepsilon,k+1}(Y) \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g\|_{L_t^2(H^1)} \right) E_{\varepsilon}(Y).
\end{aligned}$$

Summing up the above inequality with respect to k leads to (2.51). This completes the proof of Theorem 2.3.

Now let us turn to the proof of Corollary 2.1.

Proof of Corollary 2.1. By summing up (6.7) and (6.10), and then multiplying the resulting inequality by $\langle t \rangle$ and integrating the above inequality over $[0, t]$, we find

$$\begin{aligned}
& \langle t \rangle (\|X_t\|_1^2 + \|\partial_3 X\|_1^2) + \int_0^t \langle s \rangle \|\nabla X_t\|_1^2 ds \leq \|X_t\|_{L_t^2(H^1)}^2 + (1 + |\partial_3 Y|_{\frac{1}{2},1}^2) \|\partial_3 X\|_{L_t^2(H^1)}^2 \\
& + C \|\langle t \rangle^{\frac{1}{2}} |D|^{-1} g\|_{L_t^2(H^1)}^2 + C (|\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^{\frac{8}{3}} \|\partial_3 Y\|_{L_t^2(L^2)}^{\frac{4}{3}} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^4 + |Y_t|_{1+\varepsilon,2}^2) \mathcal{E}_0^2(t).
\end{aligned}$$

(2.55) then follows from (2.50).

Similarly, we get, by multiplying (6.21) by $\langle t \rangle$, then integrating the inequality over $[0, t]$ and taking the square root of the resulting inequality, that

$$\begin{aligned}
& \langle t \rangle^{\frac{1}{2}} (\|(X_t, \partial_3 X)\|_{\dot{H}^{k+1}}) + \left(\frac{3}{4} \int_0^t \langle s \rangle \|X_t\|_{\dot{H}^{k+2}}^2 ds \right)^{\frac{1}{2}} \lesssim \|\langle t \rangle^{\frac{1}{2}} g\|_{L_t^2(\dot{H}^k)} \\
& + (1 + |Y_t|_{\frac{1}{2},0}) \|X_t\|_{L_t^2(\dot{H}^{k+1})} + (1 + |\partial_3 Y|_{\frac{1}{2},0}) \|\partial_3 X\|_{L_t^2(\dot{H}^{k+1})} + |\nabla Y|_{0,k+1} \|\langle t \rangle^{\frac{1}{2}} \nabla X_t\|_{L_t^2(L^2)} \\
& + (|\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^{\frac{4}{3}} \|\partial_3 Y\|_{L_t^2(L^2)}^{\frac{2}{3}} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^2 + |Y_t|_{1+\varepsilon,1}) (\|\nabla X\|_{L_t^\infty(\dot{H}^{k+1})} + |\nabla Y|_{0,k+1} \|\nabla X\|_{L_t^\infty(H^1)}) \\
& + \|\partial_3 Y\|_{L_t^2(L^2)} (|\nabla Y|_{0,k+1} |\partial_3 Y|_{\frac{1}{2},1} + |\partial_3 Y|_{\frac{1}{2},k+1}) + (|Y_t|_{\frac{1}{2},k+1} + |\nabla Y|_{0,k+1} |Y_t|_{\frac{1}{2},1}) \|X_t\|_{L_t^2(L^2)} \\
& + (|\partial_3 Y|_{\frac{1}{2}+\varepsilon,k+1} (|\partial_3 Y|_{\frac{1}{2}+\varepsilon,0}^{\frac{1}{3}} \|\partial_3 Y\|_{L_t^2(L^2)}^{\frac{2}{3}} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon,1}^2) + |Y_t|_{1+\varepsilon,k+2}) \|\nabla X\|_{L_t^\infty(H^1)}.
\end{aligned}$$

(2.56) then follows from (2.51) and (2.55), and this completes the proof of Corollary 2.1. \square

7. ENERGY DECAY FOR ∇X_t

The main idea to prove Proposition 2.5 is to use the following proposition:

Proposition 7.1. *Let X be a smooth enough solution of*

$$(7.1) \quad \begin{cases} X_{tt} - \Delta X_t - \partial_3^2 X = \nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t) + \hbar \stackrel{\text{def}}{=} f, \\ X(0) = 0 \quad \text{and} \quad X_t(0) = 0, \end{cases}$$

on $[0, T]$. Then under the assumption that

$$(7.2) \quad \|\nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} < \delta_1,$$

we have, for any $t \in [0, T]$ and any $\varepsilon > 0$, that

$$(7.3) \quad t\|\nabla X_t(t)\|_{L^2} \leq C_\varepsilon \left(\sup_{s \in [0, t]} \|s^{1+\varepsilon}|D|^{-1}\hbar\|_{L^2} + \sup_{s \in [0, t]} \|s^{1+\varepsilon}|D|\hbar\|_{L^2} \right) \leq C_\varepsilon \| |D|^{-1}\hbar \|_{1+\varepsilon, 2}.$$

Moreover, we have for $k \in \mathbb{N}$,

$$(7.4) \quad t\|\nabla X_t(t)\|_{\dot{H}^k} \leq C_{\varepsilon, k} \left((\delta_1 + \|D^k \nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}) \| |D|^{-1}\hbar \|_{1+\varepsilon, 2} + \| |D|^{-1}\hbar \|_{1+\varepsilon, k+2} \right).$$

Admitting this proposition for the time being, we present the proof of Proposition 2.5.

Proof of Proposition 2.5. In our situation (2.40),

$$\hbar = \nabla \cdot \left(\mathcal{A}((-\nabla X)\mathcal{A} + \mathcal{A}^t(-\nabla X)^t)\mathcal{A}^t \nabla Y_t \right) - f'_1(Y; X) + f'_2(Y; X) + g.$$

We infer from (5.23), (2.46), (2.47) that for $k \geq 0$,

$$(7.5) \quad \begin{aligned} \| |D|^{-1}\hbar \|_{1+\varepsilon, k+2} &\lesssim |\partial_3 Y|_{\frac{1}{2}+\varepsilon, 0} \|\partial_3 X\|_{\frac{1}{2}, k+2} + |Y_t|_{\frac{1}{2}+\varepsilon, 0} \|X_t\|_{\frac{1}{2}, k+2} + \| |D|^{-1}g \|_{1+\varepsilon, k+2} \\ &+ (|\partial_3 Y|_{\frac{1}{2}+\varepsilon, 1}^{\frac{4}{3}} \|\partial_3 Y\|_{\frac{1}{2}, 0}^{\frac{2}{3}} + |\partial_3 Y|_{\frac{1}{2}+\varepsilon, 1}^2 + |Y_t|_{1+\varepsilon, 1}) \|\nabla X\|_{0, k+2} \\ &+ \gamma_{\varepsilon, k+2}(Y) (\|\partial_3 X\|_{\frac{1}{2}, 1} + \|X_t\|_{\frac{1}{2}, 1} + \|\nabla X\|_{0, 1}), \end{aligned}$$

where $\gamma_{\varepsilon, k+2}(Y)$ is given in (2.52). Proposition 2.5 then follows from Proposition 7.1, (7.5), Corollary 2.1 and the fact that $\|D^k \nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \leq \|\nabla Y\|_{0, k+2}$. \square

In order to prove Proposition 7.1, we need to exploit the tool of anisotropic Littlewood-Paley analysis. Similar to the dyadic operators Δ_j , and S_j given by Definition 2.1, let us recall the dyadic operators in the x_3 variable

$$(7.6) \quad \Delta_\ell^\vee a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\hat{a}), \quad \text{and} \quad S_\ell^\vee a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\hat{a}).$$

Let us also recall the following anisotropic type Besov norm from [24, 23]:

Definition 7.1. Let $s_1, s_2 \in \mathbb{R}$, $r \in [1, \infty]$ and $a \in \mathcal{S}'_h(\mathbb{R}^3)$, we define the norm

$$\|a\|_{\mathcal{B}_r^{s_1, s_2}} \stackrel{\text{def}}{=} \left(2^{js_1} 2^{\ell s_2} \|\Delta_j \Delta_\ell^\vee a\|_{L^2} \right)_{\ell^r(\mathbb{Z}^2)}.$$

In particular, when $r = 2$, we denote $\|a\|_{\dot{H}^{s_1, s_2}} \stackrel{\text{def}}{=} \|a\|_{\mathcal{B}_2^{s_1, s_2}} = \| |D|^{s_1} |D_{x_3}|^{s_2} a \|_{L^2}$.

In order to obtain a better description of the regularizing effect for the transport-diffusion equation, we will use anisotropic version of Chemin-Lerner type norm (see [3] for instance).

Definition 7.2. Let $(r, q) \in [1, +\infty]^2$ and $T \in (0, +\infty]$. We define the norm $\tilde{L}_T^q(\mathcal{B}_r^{s_1, s_2}(\mathbb{R}^3))$ by

$$\|u\|_{\tilde{L}_T^q(\mathcal{B}_r^{s_1, s_2})} \stackrel{\text{def}}{=} \left(\sum_{(j, \ell) \in \mathbb{Z}^2} (2^{js_1} 2^{\ell s_2} \|\Delta_j \Delta_\ell^\vee u\|_{L_T^q(L^2)})^r \right)^{\frac{1}{r}},$$

with the usual change if $r = \infty$.

For the convenience of the readers, we recall the following Bernstein type lemma from [3, 10, 26]:

Lemma 7.1. *Let \mathfrak{B}_h (resp. \mathfrak{B}_v) be a ball of \mathbb{R}^2 (resp. \mathbb{R}), and \mathcal{C}_h (resp. \mathcal{C}_v) a ring of \mathbb{R}^2 (resp. \mathbb{R}); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there holds:*

If the support of \hat{a} is included in $2^k \mathfrak{B}_h$, then

$$\|\partial_h^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\alpha|+2(\frac{1}{p_2}-\frac{1}{p_1}))} \|a\|_{L_h^{p_2}(L_v^{q_1})}.$$

If the support of \hat{a} is included in $2^\ell \mathfrak{B}_v$, then

$$\|\partial_3^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell(\beta+(\frac{1}{q_2}-\frac{1}{q_1}))} \|a\|_{L_h^{p_1}(L_v^{q_2})}.$$

If the support of \hat{a} is included in $2^k \mathcal{C}_h$, then

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \max_{|\alpha|=N} \|\partial_h^\alpha a\|_{L_h^{p_1}(L_v^{q_1})}.$$

If the support of \hat{a} is included in $2^\ell \mathcal{C}_v$, then

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-\ell N} \|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})}.$$

Let us now turn to the proof of Proposition 7.1.

Proof of Proposition 7.1. The proof of this lemma is motivated by the proof of Proposition 4.1 of [23, 30]. By applying the operator $\Delta_j \Delta_\ell^v$ to (7.1) and then taking the L^2 inner product of the resulting equation with $\Delta_j \Delta_\ell^v X_t$, we write

$$(7.7) \quad \frac{1}{2} \frac{d}{dt} (\|\Delta_j \Delta_\ell^v X_t\|_{L^2}^2 + \|\Delta_j \Delta_\ell^v \partial_3 X\|_{L^2}^2) + \|\nabla \Delta_j \Delta_\ell^v X_t\|_{L^2}^2 = (\Delta_j \Delta_\ell^v f | \Delta_j \Delta_\ell^v X_t)_{L^2}.$$

Along the same line, one has

$$(\Delta_j \Delta_\ell^v X_{tt} | \Delta \Delta_j \Delta_\ell^v X) - \frac{1}{2} \frac{d}{dt} \|\Delta \Delta_j \Delta_\ell^v X\|_{L^2}^2 - \|\partial_3 \nabla \Delta_j \Delta_\ell^v X\|_{L^2}^2 = (\Delta_j \Delta_\ell^v f | \Delta \Delta_j \Delta_\ell^v X).$$

Notice that

$$(\Delta_j \Delta_\ell^v X_{tt} | \Delta \Delta_j \Delta_\ell^v X) = \frac{d}{dt} (\Delta_j \Delta_\ell^v X_t | \Delta \Delta_j \Delta_\ell^v X) + \|\nabla \Delta_j \Delta_\ell^v X_t\|_{L^2}^2,$$

so that there holds

$$(7.8) \quad \frac{d}{dt} \left(\frac{1}{2} \|\Delta \Delta_j \Delta_\ell^v X\|_{L^2}^2 - (\Delta_j \Delta_\ell^v X_t | \Delta \Delta_j \Delta_\ell^v X) \right) - \|\nabla \Delta_j \Delta_\ell^v X_t\|_{L^2}^2 + \|\partial_3 \nabla \Delta_j \Delta_\ell^v X\|_{L^2}^2 = -(\Delta_j \Delta_\ell^v f | \Delta \Delta_j \Delta_\ell^v X).$$

By summing up (7.7) with $\frac{1}{4}$ of (7.8), we obtain

$$(7.9) \quad \frac{d}{dt} g_{j,\ell}^2(t) + \frac{3}{4} \|\nabla \Delta_j \Delta_\ell^v X_t\|_{L^2}^2 + \frac{1}{4} \|\partial_3 \nabla \Delta_j \Delta_\ell^v X\|_{L^2}^2 = (\Delta_j \Delta_\ell^v f | \Delta_j \Delta_\ell^v X_t - \frac{1}{4} \Delta \Delta_j \Delta_\ell^v X),$$

where

$$g_{j,\ell}^2(t) \stackrel{\text{def}}{=} \frac{1}{2} \left(\|\Delta_j \Delta_\ell^v X_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^v \partial_3 X(t)\|_{L^2}^2 + \frac{1}{4} \|\Delta_j \Delta_\ell^v \Delta X(t)\|_{L^2}^2 \right) - \frac{1}{4} (\Delta_j \Delta_\ell^v X_t(t) | \Delta_j \Delta_\ell^v \Delta X(t)).$$

It is easy to observe that

$$(7.10) \quad g_{j,\ell}^2(t) \sim \|\Delta_j \Delta_\ell^v X_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^v \partial_3 X(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^v \Delta X(t)\|_{L^2}^2.$$

Now according to the heuristic analysis presented at the beginning of Section 2, we split the frequency analysis into the following two cases:

- When $j \leq \frac{\ell+1}{2}$

In this case, one has

$$g_{j,\ell}^2(t) \sim \|\Delta_j \Delta_\ell^\vee X_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 X(t)\|_{L^2}^2,$$

and Lemma 7.1 implies that

$$\frac{3}{4} \|\nabla \Delta_j \Delta_\ell^\vee X_t\|_{L^2}^2 + \frac{1}{4} \|\partial_3 \nabla \Delta_j \Delta_\ell^\vee X\|_{L^2}^2 \geq c 2^{2j} (\|\Delta_j \Delta_\ell^\vee X_t\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 X\|_{L^2}^2).$$

Hence it follows from (7.9) that

$$\frac{d}{dt} g_{j,\ell}(t) + c 2^{2j} g_{j,\ell}(t) \leq \|\Delta_j \Delta_\ell^\vee f(t)\|_{L^2},$$

which in particular implies that

$$(7.11) \quad g_{j,\ell}(t) \leq \int_0^t e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee f(s)\|_{L^2} ds,$$

and

$$(7.12) \quad 2^j \|\Delta_j \Delta_\ell^\vee X_t\|_{L_t^1(L^2)} \lesssim 2^{-j} \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.$$

Now let us turn to the estimate of $\|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}$. Indeed it follows by the law of product in the anisotropic Besov spaces (see Lemma 3.3 of [30]) that

$$(7.13) \quad \begin{aligned} \|(\mathcal{A}\mathcal{A}^t - Id) \nabla X_t\|_{L_t^1(\dot{H}^{0,0})} &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\|_{L_t^\infty(\mathcal{B}_1^{1,\frac{1}{2}})} \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} \\ &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} \\ &\lesssim \delta_1 \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})}, \end{aligned}$$

where we used the fact that $\dot{B}_{2,1}^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \mathcal{B}_1^{1,\frac{1}{2}}$ (one may check Lemma 3.2 of [24, 30] for details). Hence we obtain

$$(7.14) \quad 2^{-j} \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)} \lesssim c_{j,\ell} \delta_1 \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + \|\Delta_j \Delta_\ell^\vee |D|^{-1} h\|_{L_t^1(L^2)}.$$

where $(c_{j,\ell})_{j,\ell \in \mathbb{Z}^2}$ is a generic element of $\ell^2(\mathbb{Z}^2)$ so that $\sum_{j,\ell \in \mathbb{Z}^2} c_{j,\ell}^2 = 1$.

Whereas it follows from Lemma 7.1 and (7.11) that

$$(7.15) \quad \begin{aligned} 2^j t \|\Delta_j \Delta_\ell^\vee X_t(t)\|_{L^2} &\lesssim \int_0^t 2^{2j} (t-s) e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee ((\mathcal{A}\mathcal{A}^t - Id) \nabla X_t)(s)\|_{L^2} ds \\ &\quad + \int_0^t 2^{2j} e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee ((\mathcal{A}\mathcal{A}^t - Id) s \nabla X_t)(s)\|_{L^2} ds \\ &\quad + t \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) 2^j e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee h(s)\|_{L^2} ds. \end{aligned}$$

By virtue of (7.13), we have

$$\begin{aligned} \int_0^t 2^{2j} (t-s) e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee ((\mathcal{A}\mathcal{A}^t - Id) \nabla X_t)(s)\|_{L^2} ds \\ \lesssim \|\Delta_j \Delta_\ell^\vee ((\mathcal{A}\mathcal{A}^t - Id) \nabla X_t)\|_{L_t^1(L^2)} \lesssim c_{j,\ell} \delta_1 \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})}. \end{aligned}$$

Along the same line, we have

$$(7.16) \quad \begin{aligned} \int_0^t 2^{2j} e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee ((\mathcal{A}\mathcal{A}^t - Id) s \nabla X_t)(s)\|_{L^2} ds \\ \lesssim \|\Delta_j \Delta_\ell^\vee ((\mathcal{A}\mathcal{A}^t - Id) s \nabla X_t)\|_{L_t^\infty(L^2)} \\ \lesssim c_{j,\ell} \|(\mathcal{A}\mathcal{A}^t - Id)\|_{L_t^\infty(\mathcal{B}_1^{1,\frac{1}{2}})} \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})} \lesssim c_{j,\ell} \delta_1 \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})}. \end{aligned}$$

While it is easy to observe from Lemma 7.1 that

$$\begin{aligned} t \int_0^{\frac{t}{2}} 2^j e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee \tilde{h}(s)\|_{L^2} ds &\lesssim \int_0^{\frac{t}{2}} (t-s) 2^{2j} e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}(s)\|_{L^2} ds \\ &\lesssim \|\Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}\|_{L_t^1(L^2)}, \end{aligned}$$

and

$$\begin{aligned} t \int_{\frac{t}{2}}^t 2^j e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee \tilde{h}(s)\|_{L^2} ds &\lesssim \int_{\frac{t}{2}}^t \langle t-s \rangle^{-1} (2^{-j} + 2^j) \|s \Delta_j \Delta_\ell^\vee \tilde{h}(s)\|_{L^2} ds \\ &\lesssim \int_{\frac{t}{2}}^t \langle t-s \rangle^{-1} (\|s \Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}(s)\|_{L^2} + \|s \Delta_j \Delta_\ell^\vee |D| \tilde{h}(s)\|_{L^2}) ds. \end{aligned}$$

Substituting the above estimates into (7.15) leads to

$$\begin{aligned} (7.17) \quad 2^j t \|\Delta_j \Delta_\ell^\vee X_t(t)\|_{L^2} &\lesssim c_{j,\ell} \delta_1 (\|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})}) + \|\Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}\|_{L_t^1(L^2)} \\ &\quad + \int_{\frac{t}{2}}^t \langle t-s \rangle^{-1} (\|s \Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}(s)\|_{L^2} + \|s \Delta_j \Delta_\ell^\vee |D| \tilde{h}(s)\|_{L^2}) ds \end{aligned}$$

for all (j, ℓ) satisfying $j \leq \frac{\ell+1}{2}$.

• When $j > \frac{\ell+1}{2}$

In this case, we have

$$g_{j,\ell}^2(t) \sim \|\Delta_j \Delta_\ell^\vee X_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \Delta X(t)\|_{L^2}^2$$

and Lemma 7.1 implies that

$$\begin{aligned} \frac{3}{4} \|\nabla \Delta_j \Delta_\ell^\vee X_t\|_{L^2}^2 + \frac{1}{4} \|\partial_3 \nabla \Delta_j \Delta_\ell^\vee X\|_{L^2}^2 &\geq c(2^{2j} \|\Delta_j \Delta_\ell^\vee X_t\|_{L^2}^2 + 2^{2j} 2^{2\ell} \|\Delta_j \Delta_\ell^\vee X\|_{L^2}^2) \\ &\geq c \frac{2^{2\ell}}{2^{2j}} (\|\Delta_j \Delta_\ell^\vee X_t\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \Delta X\|_{L^2}^2). \end{aligned}$$

Then we deduce from (7.9) that

$$\frac{d}{dt} g_{j,\ell}(t) + c 2^{2(\ell-j)} g_{j,\ell}(t) \leq \|\Delta_j \Delta_\ell^\vee f(t)\|_{L^2},$$

which implies that

$$(7.18) \quad g_{j,\ell}(t) \leq \int_0^t e^{-c(t-s)2^{2(\ell-j)}} \|\Delta_j \Delta_\ell^\vee f(s)\|_{L^2} ds,$$

and

$$(7.19) \quad 2^{2\ell} \|\Delta_j \Delta_\ell^\vee X\|_{L_t^1(L^2)} \lesssim \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.$$

On the other hand, we get, by taking L^2 inner product of (7.1) with $\Delta_j \Delta_\ell^\vee X_t$, that

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j \Delta_\ell^\vee X_t\|_{L^2}^2 + \|\nabla \Delta_j \Delta_\ell^\vee X_t\|_{L^2}^2 = (\partial_3^2 \Delta_j \Delta_\ell^\vee X + \Delta_j \Delta_\ell^\vee f \mid \Delta_j \Delta_\ell^\vee X_t)_{L^2},$$

from which, Lemma 7.1, we infer

$$\frac{d}{dt} \|\Delta_j \Delta_\ell^\vee X_t(t)\|_{L^2} + c 2^{2j} \|\Delta_j \Delta_\ell^\vee X_t(t)\|_{L^2} \lesssim 2^{2\ell} \|\Delta_j \Delta_\ell^\vee X(t)\|_{L^2} + \|\Delta_j \Delta_\ell^\vee f(t)\|_{L^2},$$

so that there hold

$$\begin{aligned} (7.20) \quad 2^j \|\Delta_j \Delta_\ell^\vee X_t(t)\|_{L^2} &\lesssim 2^{2\ell+j} \int_0^t e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee X(s)\|_{L^2} ds \\ &\quad + 2^j \int_0^t e^{-c(t-s)2^{2j}} \|\Delta_j \Delta_\ell^\vee f(s)\|_{L^2} ds. \end{aligned}$$

And then we deduce from (7.19) that for $j > \frac{\ell+1}{2}$

$$(7.21) \quad \begin{aligned} 2^j \|\Delta_j \Delta_\ell^\vee X_t\|_{L_t^1(L^2)} &\lesssim 2^{2\ell-j} \|\Delta_j \Delta_\ell^\vee X\|_{L_t^1(L^2)} + 2^{-j} \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)} \\ &\lesssim 2^{-j} \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}. \end{aligned}$$

Moreover, in this case, it follows from Lemma 7.1 and (7.18) that

$$\begin{aligned} 2^{2\ell+j} t \int_0^t e^{-c2^{2j}(t-s)} \|\Delta_j \Delta_\ell^\vee X(s)\|_{L^2} ds &\lesssim 2^{2\ell-j} t \|\Delta_j \Delta_\ell^\vee X\|_{L_t^\infty(L^2)} \\ &\lesssim 2^{2\ell-3j} t \|\Delta_j \Delta_\ell^\vee \Delta X\|_{L_t^\infty(L^2)} \lesssim 2^{2\ell-3j} t \|g_{j,\ell}\|_{L_t^\infty} \\ &\lesssim 2^{2\ell-3j} t \int_0^t e^{-c(t-s)2^{2(\ell-j)}} \|\Delta_j \Delta_\ell^\vee f(s)\|_{L^2} ds, \end{aligned}$$

from which and a similar proof of (7.17), we infer

$$(7.22) \quad \begin{aligned} 2^{2\ell+j} t \int_0^t e^{-c2^{2j}(t-s)} \|\Delta_j \Delta_\ell^\vee X(s)\|_{L^2} ds &\lesssim c_{j,\ell} \delta_1 (\|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})}) \\ &\quad + \|\Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}\|_{L_t^1(L^2)} + \int_{\frac{t}{2}}^t \langle t-s \rangle^{-1} (\|s \Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}(s)\|_{L^2} + \|s \Delta_j \Delta_\ell^\vee |D| \tilde{h}(s)\|_{L^2}) ds. \end{aligned}$$

Here we used the fact $j \geq \ell - N_0$ for some fixed integer N_0 in the operator $\Delta_j \Delta_\ell^\vee$.

By virtue of (7.20) and (7.22), we get, by a similar derivation of (7.17) that (7.17) holds for all $(j, \ell) \in \mathbb{Z}^2$. Furthermore, in view of (7.12)-(7.21), we obtain for all $(j, \ell) \in \mathbb{Z}^2$, that

$$(7.23) \quad 2^j \|\Delta_j \Delta_\ell^\vee X_t\|_{L_t^1(L^2)} \lesssim 2^{-j} \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.$$

Inserting (7.14) into (7.23) gives rise to

$$\begin{aligned} \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} &= \left(\sum_{j,\ell \in \mathbb{Z}^2} 2^{2j} \|\Delta_j \Delta_\ell^\vee X_t\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} \\ &\leq C \delta_1 \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + C \left(\sum_{j,\ell \in \mathbb{Z}^2} \|\Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}\|_{L_t^1(L^2)}^2 \right)^{\frac{1}{2}} \\ &\leq C \delta_1 \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + C \int_0^t \left(\sum_{j,\ell \in \mathbb{Z}^2} \|\Delta_j \Delta_\ell^\vee |D|^{-1} \tilde{h}(s)\|_{L^2}^2 \right)^{\frac{1}{2}} ds \\ &\leq C (\delta_1 \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + \||D|^{-1} \tilde{h}\|_{L_t^1(L^2)}). \end{aligned}$$

In particular, by taking δ_1 to be sufficiently small in (7.2), we conclude that

$$(7.24) \quad \|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} \leq C \||D|^{-1} \tilde{h}\|_{L_t^1(L^2)}.$$

Along the same line, we deduce from (7.17) that

$$(7.25) \quad \begin{aligned} \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})} &= \left(\sum_{j,\ell \in \mathbb{Z}^2} 2^{2j} \|t \Delta_j \Delta_\ell^\vee X_t\|_{L_t^\infty(L^2)}^2 \right)^{\frac{1}{2}} \\ &\leq C \left(\delta_1 (\|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})}) + \||D|^{-1} \tilde{h}\|_{L_t^1(L^2)} \right. \\ &\quad \left. + \int_{\frac{t}{2}}^t \langle t-s \rangle^{-1} (\|s |D|^{-1} \tilde{h}(s)\|_{L^2} + \|s |D| \tilde{h}(s)\|_{L^2}) ds \right). \end{aligned}$$

So that by taking δ_1 is small enough in (7.2), we obtain

$$\begin{aligned}
 (7.26) \quad & t \|\nabla X_t(t)\|_{L^2} \leq \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})} \\
 & \leq C \left(\| |D|^{-1} \tilde{h} \|_{L_t^1(L^2)} + \int_{\frac{t}{2}}^t \langle t-s \rangle^{-1} (\|s |D|^{-1} \tilde{h}(s)\|_{L^2} + \|s |D| \tilde{h}(s)\|_{L^2}) ds \right) \\
 & \leq C_\varepsilon \left(\sup_{s \in [0,t]} \|s^{1+\varepsilon} |D|^{-1} \tilde{h}\|_{L^2} + \sup_{s \in [0,t]} \|s^{1+\varepsilon} |D| \tilde{h}\|_{L^2} \right),
 \end{aligned}$$

which leads to (7.3).

The proof of the general estimates (7.4) follows along the same line. Indeed for any $k \geq 1$, we have

$$\begin{aligned}
 \|D^k((\mathcal{A}\mathcal{A}^t - Id)\nabla X_t)\|_{\tilde{L}_t^1(\dot{H}^{0,0})} & \lesssim C_k \sum_{k_1+k_2=k} \|D^{k_1}(\mathcal{A}\mathcal{A}^t - Id)\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \|D^{k_2}\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} \\
 & \lesssim C_k \sum_{k_1+k_2=k} \|D^{k_1}\nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \|D^{k_2}\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})},
 \end{aligned}$$

from which and a similar derivation of (7.24), we inductively infer that

$$\begin{aligned}
 \|D^k \nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} & \leq C \| |D|^{k-1} \tilde{h} \|_{L_t^1(L^2)} \\
 & \quad + C_k \sum_{k_1+\dots+k_\ell=k} \|D^{k_1}\nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \cdots \|D^{k_\ell}\nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \| |D|^{-1} \tilde{h} \|_{L_t^1(L^2)}.
 \end{aligned}$$

Hence by applying the interpolation inequality that

$$\|D^{k_i}\nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})} \lesssim \|\nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}^{1-k_i/k} \|D^k \nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}^{k_i/k} \quad \text{for } k_i > 1,$$

and the assumption (7.2), we obtain

$$(7.27) \quad \|D^k \nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} \leq C_k \left((\delta_1 + \|D^k \nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}) \| |D|^{-1} \tilde{h} \|_{L_t^1(L^2)} + \| |D|^{k-1} \tilde{h} \|_{L_t^1(L^2)} \right).$$

While it follows from a similar derivation of (7.25) that

$$\begin{aligned}
 \|t D^k \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})} & \leq C_k \left(\| |D|^{k-1} \tilde{h} \|_{L_t^1(L^2)} + \delta_1 (\|D^k \nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + \|t D^k \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})}) \right. \\
 & \quad \left. + (\delta_1 + \|D^k \nabla Y\|_{L_t^\infty(\dot{B}_{2,1}^{\frac{3}{2}})}) (\|\nabla X_t\|_{\tilde{L}_t^1(\dot{H}^{0,0})} + \|t \nabla X_t\|_{\tilde{L}_t^\infty(\dot{H}^{0,0})}) \right. \\
 & \quad \left. + \int_{\frac{t}{2}}^t \langle t-s \rangle^{-1} (\|s |D|^{k-1} \tilde{h}(s)\|_{L^2} + \|s |D|^{k+1} \tilde{h}(s)\|_{L^2}) ds \right).
 \end{aligned}$$

Thus (7.4) follows (7.27) and the argument in (7.26). This completes the proof of Proposition 7.1. \square

8. ESTIMATES OF THE SOURCE TERM $f(Y)$

In this section, we shall present the estimates to the nonlinear source term $f(Y)$ determined by (2.25).

• The estimate of $\|f(Y)\|_{\delta, N}$

Proposition 8.1. *Let the functionals f_0, f_1, f_2 be given in (4.3) and the norm $\|\cdot\|_{\delta, N}$ by (2.39). Then under the assumption of (2.41), we have*

$$(8.1) \quad \|f_0(Y)\|_{\delta, N} \lesssim \|\nabla Y\|_0 \|\nabla Y_t\|_{N+6} + \|\nabla Y\|_{N+6} \|\nabla Y_t\|_0;$$

$$(8.2) \quad \|f_1(Y)\|_{\delta, N} \lesssim \|\partial_3 Y\|_0 \|\partial_3 Y\|_{N+6} + \|\nabla Y\|_{N+6} |\partial_3 Y|_0 \|\partial_3 Y\|_1;$$

$$(8.3) \quad \|f_2(Y)\|_{\delta, N} \lesssim \|Y_t\|_0 \|Y_t\|_{N+6} + \|\nabla Y\|_{N+6} |Y_t|_0 \|Y_t\|_1.$$

Proof. As in Section 4, we shall deal with the estimate of $f(Y)$ by the norm of the homogeneous Besov space $\dot{B}_{1,1}^s$ instead of the one in the homogeneous Sobolev space $\dot{W}^{s,1}$. Indeed in view of (4.3), we get, by applying the law of product, (5.1), that for $s > 0$,

$$\|f_0(Y)\|_{\dot{B}_{1,1}^s} \lesssim \|(\mathcal{A}^t \mathcal{A} - Id) \nabla Y_t\|_{\dot{B}_{1,1}^{s+1}} \lesssim \|\nabla Y\|_0 \|\nabla Y_t\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} \|\nabla Y_t\|_0.$$

(8.1) then follows from the above inequality and the interpolation inequality (5.22). Along the same line, we deduce from (4.3) that

$$\begin{aligned} \|f_m(Y)\|_{\dot{B}_{1,1}^s} &\lesssim (1 + |\mathcal{A}^t - Id|_0) \|\nabla \mathbf{p}_m\|_{\dot{B}_{1,1}^s} + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^s} \|\nabla \mathbf{p}_m\|_0 \\ &\lesssim \|\nabla \mathbf{p}_m\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla \mathbf{p}_m\|_0. \end{aligned}$$

Yet it follows from (4.1) that

$$\begin{aligned} \|\nabla \mathbf{p}_1\|_{\dot{B}_{1,1}^s} &\lesssim \delta_1 \|\nabla \mathbf{p}_1\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla \mathbf{p}_1\|_0 + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{B}_{1,1}^{s+1}} \\ &\quad + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{\dot{H}^1}, \end{aligned}$$

which together with (5.6) implies

$$\|\nabla \mathbf{p}_1\|_{\dot{B}_{1,1}^s} \lesssim \|\partial_3 Y\|_0 \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^s \cap \dot{B}_{2,1}^{s+1}} \|\partial_3 Y\|_0 \|\partial_3 Y\|_1.$$

As a result, it comes out

$$\|f_1(Y)\|_{\dot{B}_{1,1}^s} \lesssim \|\partial_3 Y\|_0 \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^s \cap \dot{B}_{2,1}^{s+1}} \|\partial_3 Y\|_0 \|\partial_3 Y\|_1.$$

Similarly, we have

$$\|f_2(Y)\|_{\dot{B}_{1,1}^s} \lesssim \|Y_t\|_0 \|Y_t\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla Y\|_{\dot{B}_{2,1}^s \cap \dot{B}_{2,1}^{s+1}} \|Y_t\|_0 \|Y_t\|_1.$$

(8.2) and (8.3) then follow from the above estimates and the interpolation inequality (5.22). This completes the proof of Proposition 8.1. \square

• The estimate of $\| |D|^{-1} f(Y) \|_{N+1}$

Proposition 8.2. *Under the asme assumptions of Proposition 8.1, we have*

$$(8.4) \quad \| |D|^{-1} f_0(Y) \|_{N+1} \lesssim |\nabla Y|_0 \|\nabla Y_t\|_{N+1} + |\nabla Y|_{N+1} \|\nabla Y_t\|_0;$$

$$(8.5) \quad \| |D|^{-1} f_1(Y) \|_{N+1} \lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_{N+1} + |\nabla Y|_{N+1} \|\partial_3 Y\|_0 \|\partial_3 Y\|_1;$$

$$(8.6) \quad \| |D|^{-1} f_2(Y) \|_{N+1} \lesssim |Y_t|_0 \|Y_t\|_{N+1} + |\nabla Y|_{N+1} \|Y_t\|_0 \|Y_t\|_1.$$

Proof. In view of (4.3), we get, by applying Moser type inequality, that

$$\| |D|^{-1} f_0(Y) \|_N \leq \|(\mathcal{A}^t \mathcal{A} - Id) \nabla Y_t\|_N \lesssim |\nabla Y|_0 \|\nabla Y_t\|_N + |\nabla Y|_N \|\nabla Y_t\|_0,$$

which gives (8.4). While again by (4.3) and the law of product in Besov spaces, one has

$$\| |D|^{-1} f_m(Y) \|_0 \lesssim (1 + \|\mathcal{A}^t - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \|\nabla \mathbf{p}_m\|_{\dot{H}^{-1}},$$

yet it follows from (4.1) that

$$\|\nabla \mathbf{p}_1\|_{\dot{H}^{-1}} \lesssim \|\nabla Y\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\nabla \mathbf{p}_1\|_{\dot{H}^{-1}} + (1 + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_0,$$

from which and the assumption (2.41), we infer

$$(8.7) \quad \| |D|^{-1} f_1(Y) \|_0 \lesssim \|\nabla \mathbf{p}_1\|_{\dot{H}^{-1}} \lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_0.$$

Similarly, we have

$$(8.8) \quad \| |D|^{-1} f_2(Y) \|_0 \lesssim \|\nabla \mathbf{p}_2\|_{\dot{H}^{-1}} \lesssim |Y_t|_0 \|Y_t\|_0.$$

For $N \geq 0$, we deduce from (4.3) that

$$\|f_1(Y)\|_N \lesssim \|\nabla \mathbf{p}_1\|_N + |\nabla Y|_N \|\nabla \mathbf{p}_1\|_0.$$

And it follows from (4.1) that

$$\|\nabla \mathbf{p}_1\|_N \lesssim |\nabla Y|_0 \|\nabla \mathbf{p}_1\|_N + |\nabla Y|_N \|\nabla \mathbf{p}_1\|_0 + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y))\|_N,$$

which together with (2.41) and (5.6) ensures that

$$\|\nabla \mathbf{p}_1\|_N \lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_{N+1} + |\nabla Y|_{N+1} |\partial_3 Y|_0 \|\partial_3 Y\|_1.$$

As a result, it comes out

$$(8.9) \quad \|f_1(Y)\|_N \lesssim |\partial_3 Y|_0 \|\partial_3 Y\|_{N+1} + |\nabla Y|_{N+1} |\partial_3 Y|_0 \|\partial_3 Y\|_1.$$

The same procedure for $f_2(Y)$ yields

$$(8.10) \quad \|f_2(Y)\|_N \lesssim |Y_t|_0 \|Y_t\|_{N+1} + |\nabla Y|_{N+1} |Y_t|_0 \|Y_t\|_1.$$

(8.5) and (8.6) follow from (8.7)-(8.10). This completes the proof of Proposition 8.2. \square

9. ESTIMATES OF $f''(Y; X, W)$

The purpose of this section is to present the related estimates to the second derivatives, $f''(Y; X, W)$, of the nonlinear functional $f(Y)$ given by (2.25).

9.1. The estimate of $\| |D|^{-1} f''(Y; X, W) \|_N$.

Proposition 9.1. *Let f''_0, f''_1, f''_2 be given by (4.13) and (4.14) respectively. Then under the assumption of (2.41), we have*

$$(9.1) \quad \begin{aligned} \| |D|^{-1} f''_0(Y; X, W) \|_N &\lesssim |Y_t|_1 (|\nabla X|_N \|\nabla W\|_0 + |\nabla X|_0 \|\nabla W\|_N) \\ &+ ((|Y_t|_{N+1} + |\nabla Y|_N |Y_t|_1) |\nabla X|_0 + |X_t|_{N+1}) \|\nabla W\|_0 + |\nabla X|_N \|\nabla W_t\|_0 \\ &+ |\nabla X|_0 \|\nabla W_t\|_N + |\nabla Y|_N (|\nabla X|_0 \|\nabla W_t\|_0 + |X_t|_1 \|\nabla W\|_0) + |X_t|_1 \|\nabla W\|_N, \end{aligned}$$

and

$$(9.2) \quad \| |D|^{-1} f''_1(Y; X, W) \|_N \lesssim \mathfrak{f}_3(\partial_3 Y, \partial_3 X, \partial_3 W) \quad \text{and}$$

$$(9.3) \quad \| |D|^{-1} f''_2(Y; X, W) \|_N \lesssim \mathfrak{f}_3(Y_t, X_t, W_t),$$

where the functional $\mathfrak{f}_3(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ is given by

$$\begin{aligned} \mathfrak{f}_3(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) &\stackrel{\text{def}}{=} (|\mathfrak{y}|_N + |\nabla Y|_N |\mathfrak{y}|_0) \|\mathfrak{z}\|_0 + |\mathfrak{y}|_0 \|\mathfrak{z}\|_N \\ &+ (|\mathfrak{x}|_0 + |\mathfrak{x}|_1^{\frac{1}{3}} \|\mathfrak{x}\|_1^{\frac{2}{3}}) (|\nabla X|_N \|\mathfrak{z}\|_0 + |\nabla X|_0 \|\mathfrak{z}\|_N + |\mathfrak{y}|_1 \|\nabla W\|_N \\ &+ (|\mathfrak{y}|_N + |\mathfrak{x}|_N |\nabla X|_1) \|\nabla W\|_1) + (|\mathfrak{x}|_N + |\nabla Y|_N |\mathfrak{x}|_1) |\nabla X|_1 \|\mathfrak{z}\|_1 \\ &+ (|\mathfrak{x}|_1^{\frac{4}{3}} \|\mathfrak{x}\|_0^{\frac{2}{3}} + |\mathfrak{x}|_1^2) ((|\nabla X|_N + |\nabla Y|_N |\nabla X|_1) \|\nabla W\|_1 + |\nabla X|_1 \|\nabla W\|_N) \\ &+ (|\mathfrak{x}|_N + |\mathfrak{x}|_N^{\frac{1}{3}} \|\mathfrak{x}\|_N^{\frac{2}{3}} + |\nabla Y|_N (|\mathfrak{x}|_1 + |\mathfrak{x}|_0^{\frac{1}{3}} \|\mathfrak{x}\|_0^{\frac{2}{3}})) |\mathfrak{y}|_1 \|\nabla W\|_1. \end{aligned}$$

Remark 9.1. *We mention that in the above inequalities, it is crucial to estimate the vector, X , by L^∞ -norm. In Section 10, we shall deal with the estimate of the error term*

$$e'_p = - \int_0^1 f''(Y_p + s(1 - S_p)Y_p; (1 - S_p)Y_p, X_p) ds,$$

where the variable, $(1 - S_p)Y_p$, is “small” in the L^∞ -norm, but only “bounded” in L^2 -norm.

Let us start the proof of Proposition 9.1 by the following lemma:

Lemma 9.1. *Under the assumption of (2.41), one has*

$$\begin{aligned} \|\mathcal{A}'(Y; X)\|_0 &\lesssim \|\nabla X\|_0 & \text{and} & \quad |\mathcal{A}'(Y; X)|_N \lesssim |\nabla X|_N + |\nabla Y|_N |\nabla X|_0; \\ \|(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\|_0 &\lesssim \|\nabla X\|_0 & \text{and} & \quad |(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)|_N \lesssim |\nabla X|_N + |\nabla Y|_N |\nabla X|_0; \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{A}''(Y; X, W)\|_N + \|(\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)\|_N \\ \leq |\nabla X|_N \|\nabla W\|_0 + |\nabla X|_0 (\|\nabla W\|_N + |\nabla Y|_N \|\nabla W\|_0). \end{aligned}$$

Proof. Indeed the estimates for $\mathcal{A}'(Y; X)$, $(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)$ and $\mathcal{A}''(Y; X, W)$ can be deduced by applying Moser type inequality to (4.4), (4.5) and (4.11). Whereas in view of (4.10), we have

$$\begin{aligned} (\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W) &= \mathcal{A}''(Y; X, W)\mathcal{A}^t + \mathcal{A}(\mathcal{A}^t)''(Y; X, W) \\ (9.4) \quad &+ \mathcal{A}'(Y; X)(\mathcal{A}^t)'(Y; W) + \mathcal{A}'(Y; W)(\mathcal{A}^t)'(Y; X). \end{aligned}$$

Thus the estimate for $(\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)$ follows. \square

Proof of Proposition 9.1. We divide the proof of this proposition into the following steps.

• The estimate of $\| |D|^{-1} f_0''(Y; X, W) \|_N$

We first deduce from Moser type inequality and Lemma 9.1 that

$$\begin{aligned} \|(\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)\nabla Y_t\|_N &\lesssim |\nabla Y_t|_N |\nabla X|_0 \|\nabla W\|_0 \\ &+ |\nabla Y_t|_0 (|\nabla X|_N \|\nabla W\|_0 + |\nabla X|_0 (\|\nabla W\|_N + |\nabla Y|_N \|\nabla W\|_0)), \end{aligned}$$

and

$$\|(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\nabla W_t\|_N \lesssim |\nabla X|_N \|\nabla W_t\|_0 + |\nabla X|_0 (\|\nabla W_t\|_N + |\nabla Y|_N \|\nabla W_t\|_0),$$

and

$$\|(\mathcal{A}\mathcal{A}^t - Id)'(Y; W)\nabla X_t\|_N \lesssim |X_t|_{N+1} \|\nabla W\|_0 + |X_t|_1 (\|\nabla W\|_N + |\nabla Y|_N \|\nabla W\|_0).$$

Hence thanks to (4.13), we obtain (9.1).

Next, we shall only present the estimates for f_1'' , the one for f_2'' follows along the same line. According to (4.14), we write

$$\begin{aligned} f_1''(Y; X, W) &= I_1 + I_2 + I_3 + I_4 \quad \text{with} \\ (9.5) \quad I_1 &\stackrel{\text{def}}{=} (\mathcal{A}^t)''(Y; X, W)\nabla \mathbf{p}_1(Y), & I_2 &\stackrel{\text{def}}{=} (\mathcal{A}^t)'(Y; X)\nabla(\mathbf{p}_1'(Y; W)), \\ I_3 &\stackrel{\text{def}}{=} (\mathcal{A}^t)'(Y; W)\nabla(\mathbf{p}_1'(Y; X)), & \text{and} & \quad I_4 \stackrel{\text{def}}{=} \mathcal{A}^t \nabla(\mathbf{p}_1''(Y; X, W)). \end{aligned}$$

• The estimate of $\|f_1''(Y; X, W)\|_{\dot{H}^{-1}}$

(i) \dot{H}^{-1} estimate of I_1 . By virtue of Sobolev embedding: $L^{\frac{6}{5}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-1}(\mathbb{R}^3)$, and (5.13), we infer

$$\|I_1\|_{\dot{H}^{-1}} \lesssim \|(\mathcal{A}^t)''(Y; X, W)\|_0 \|\nabla \mathbf{p}_1\|_{L^3} \lesssim |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_0 \|\nabla W\|_0.$$

(ii) \dot{H}^{-1} estimate of I_2 . Similar to the estimate of I_1 , we have

$$\|I_2\|_{\dot{H}^{-1}} \lesssim |(\mathcal{A}^t)'(Y; X)|_0 \|\nabla \mathbf{p}_1'(Y; W)\|_{L^{\frac{6}{5}}}.$$

Yet it follows from (4.8) that

$$\begin{aligned} \|\nabla \mathbf{p}_1'(Y; W)\|_{L^{\frac{6}{5}}} &\lesssim \delta_1 \|\nabla \mathbf{p}_1'(Y; W)\|_{L^{\frac{6}{5}}} + \|(\mathcal{A}\mathcal{A}^t - Id)'(Y; W)\|_0 \|\nabla \mathbf{p}_1\|_{L^3} \\ &+ \|\mathcal{A}'(Y; W)\|_1 \|\partial_3 Y \otimes \partial_3 Y\|_{W^{1,3}} + \|\mathcal{A}'(Y; W)\|_0 \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}} \\ &+ \|\mathcal{A}(\partial_3 Y \otimes \partial_3 W + \partial_3 W \otimes \partial_3 Y)\|_{W^{1, \frac{6}{5}}}, \end{aligned}$$

which together with (5.13) and Lemma 9.1 implies that

$$(9.6) \quad \|\nabla \mathbf{p}'_1(Y; W)\|_{L^{\frac{6}{5}}} \lesssim |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} \|\nabla W\|_1 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} \|\partial_3 W\|_1.$$

As a result, it comes out

$$\|I_2\|_{\dot{H}^{-1}} \lesssim (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} \|\nabla W\|_1 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} \|\partial_3 W\|_1) |\nabla X|_0.$$

(iii) \dot{H}^{-1} estimate of I_3 . Note that

$$\|I_3\|_{\dot{H}^{-1}} \lesssim \|(\mathcal{A}^t)'(Y; W)\|_0 \|\nabla(\mathbf{p}'_1(Y; X))\|_{L^3}.$$

While in view of (4.8), one has

$$\begin{aligned} \|\nabla \mathbf{p}'_1(Y; X)\|_{L^3} &\lesssim \delta_1 \|\nabla \mathbf{p}'_1(Y; X)\|_{L^3} + |(\mathcal{A} \mathcal{A}^t - Id)'(Y; X)|_0 \|\nabla \mathbf{p}_1\|_{L^3} \\ &\quad + \|\mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}} + |\mathcal{A}'(Y; X)|_0 \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}} \\ &\quad + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)\|_{W^{1,3}}, \end{aligned}$$

which together with (5.13) and Lemma 9.1 ensures that

$$(9.7) \quad \|\nabla \mathbf{p}'_1(Y; X)\|_{L^3} \lesssim |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} |\partial_3 X|_1.$$

And we thus obtain

$$\|I_3\|_{\dot{H}^{-1}} \lesssim (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} |\partial_3 X|_1) \|\nabla W\|_0.$$

(iv) \dot{H}^{-1} estimate of I_4 . We first get, by applying the law of product in Besov spaces, that

$$\|I_4\|_{\dot{H}^{-1}} \lesssim (1 + \|\mathcal{A}^t - Id\|_{\dot{B}_{2,1}^{\frac{3}{2}}}) \|\mathbf{p}''_m(Y; X, W)\|_0.$$

Thanks to (4.15), we get, by applying Sobolev embedding: $L^{\frac{6}{5}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-1}(\mathbb{R}^3)$, that

$$\begin{aligned} \|\mathbf{p}''_1(Y; X, W)\|_0 &\lesssim \|\nabla Y\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|\mathbf{p}'_1(Y; X, W)\|_0 + \|(\mathcal{A} \mathcal{A}^t - Id)''(Y; X, W)\|_0 \|\nabla \mathbf{p}_1\|_{L^3} \\ &\quad + |(\mathcal{A} \mathcal{A}^t - Id)'(Y; X)|_0 \|\nabla \mathbf{p}'_1(Y; W)\|_{L^{\frac{6}{5}}} + \|(\mathcal{A} \mathcal{A}^t - Id)'(Y; W)\|_0 \|\nabla \mathbf{p}'_1(Y; X)\|_{L^3} \\ &\quad + \|\mathcal{A}(\partial_3 X \otimes \partial_3 W + \partial_3 W \otimes \partial_3 X)\|_0 + \|\mathcal{A}''(Y; X, W)\|_0 (|\partial_3 Y|_0^2 + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}}) \\ &\quad + |\mathcal{A}'(Y; X)|_0 (\|(\partial_3 Y \otimes \partial_3 W + \partial_3 W \otimes \partial_3 Y)\|_0 + \|\mathcal{A}'(Y; W)(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1, \frac{6}{5}}}) \\ &\quad + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 W + \partial_3 W \otimes \partial_3 Y)\|_{W^{1, \frac{6}{5}}} + |\mathcal{A}'(Y; W)|_0 \|(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)\|_0 \\ &\quad + \|\mathcal{A}'(Y; W)\|_0 (\|\mathcal{A}(\partial_3 Y \otimes \partial_3 X + \partial_3 X \otimes \partial_3 Y)\|_{W^{1,3}} + \|\mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}}), \end{aligned}$$

from which, (2.41), (5.13), Lemma 9.1, (9.6) and (9.7), we infer

$$(9.8) \quad \begin{aligned} \|\mathbf{p}''_1(Y; X, W)\|_0 &\lesssim |\partial_3 X|_0 \|\partial_3 W\|_0 + |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1 \|\nabla W\|_1 \\ &\quad + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} (|\nabla X|_0 \|\partial_3 W\|_1 + |\partial_3 X|_1 \|\nabla W\|_1). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|I_4\|_{\dot{H}^{-1}} &\lesssim |\partial_3 X|_0 \|\partial_3 W\|_0 + |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1 \|\nabla W\|_1 \\ &\quad + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} (|\nabla X|_0 \|\partial_3 W\|_1 + |\partial_3 X|_1 \|\nabla W\|_1). \end{aligned}$$

By summarizing the estimates of I_1, I_2, I_3, I_4 , we achieve

$$(9.9) \quad \begin{aligned} \|f''_1(Y; X, W)\|_{\dot{H}^{-1}} &\lesssim |\partial_3 X|_0 \|\partial_3 W\|_0 + |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1 \|\nabla W\|_1 \\ &\quad + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} (|\nabla X|_0 \|\partial_3 W\|_1 + |\partial_3 X|_1 \|\nabla W\|_1). \end{aligned}$$

• The estimate of $\|f''_m(Y; X, W)\|_N$

(i) H^N estimate of I_1 . In view of (9.5), we deduce from (5.13), (5.33) and Lemma 9.1 that

$$\begin{aligned} \|I_1\|_N &\lesssim \|(\mathcal{A}^t)''(Y; X, W)\|_{N+1} \|\nabla \mathbf{p}_m\|_{L^3} + \|(\mathcal{A}^t)''(Y; X, W)\|_1 \|\nabla \mathbf{p}_m\|_{W^{N,3}} \\ &\lesssim |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} (|\nabla X|_{N+1} \|\nabla W\|_0 + |\nabla X|_0 \|\nabla W\|_{N+1}) \\ &\quad + (|\partial_3 Y|_{N+1} |\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{N+1} |\partial_3 Y|_1^{\frac{4}{3}}) \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1 \|\nabla W\|_1. \end{aligned}$$

(ii) H^N estimate of I_2 . By virtue of (5.36), we get, by applying Moser type inequality, that

$$\begin{aligned} \|I_2\|_N &\lesssim |\partial_3 Y|_0 (|\nabla X|_{N+1} \|\partial_3 W\|_0 + |\nabla X|_0 \|\partial_3 W\|_{N+1}) \\ &\quad + (|\nabla X|_{N+1} \|\nabla W\|_0 + |\nabla X|_0 \|\nabla W\|_{N+1} + |\nabla Y|_{N+1} |\nabla X|_0 \|\nabla W\|_1) \times \\ &\quad \times (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) + (|\partial_3 Y|_{N+1} + |\nabla Y|_{N+1} |\partial_3 Y|_1) |\nabla X|_0 \|\partial_3 W\|_1 \\ &\quad + |\nabla X|_0 \|\nabla W\|_1 |\partial_3 Y|_{N+1} (|\partial_3 Y|_0^{\frac{1}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_0). \end{aligned}$$

(iii) H^N estimate of I_3 . Applying Moser type inequality gives

$$\|I_3\|_N \lesssim \|(\mathcal{A}^t)'(Y; W)\|_1 \|\nabla \mathbf{p}'_1(Y; X)\|_{W^{N,3}} + \|(\mathcal{A}^t)'(Y; W)\|_{N+1} \|\nabla \mathbf{p}'_1(Y; X)\|_{L^3}.$$

Yet it follows from (4.8) that

$$\begin{aligned} \|\nabla \mathbf{p}'_1(Y; X)\|_{W^{N,3}} &\lesssim \delta_1 \|\nabla \mathbf{p}'_1(Y; X)\|_{W^{N,3}} + |\nabla Y|_N \|\nabla \mathbf{p}'_1(Y; X)\|_{L^3} \\ &\quad + |(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)|_N \|\nabla \mathbf{p}_1\|_{L^3} + |(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)|_0 \|\nabla \mathbf{p}_1\|_{W^{N,3}} \\ &\quad + \|\mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 Y)\|_{W^{N+1,3}} + |\nabla Y|_N \|\mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}} \\ &\quad + |\mathcal{A}'(Y; X)|_N \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{W^{1,3}} + |\mathcal{A}'(Y; X)|_0 \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y)\|_{W^{N+1,3}} \\ &\quad + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 X)\|_{W^{N+1,3}} + |\nabla Y|_N \|\mathcal{A}(\partial_3 Y \otimes \partial_3 X)\|_{W^{1,3}}, \end{aligned}$$

from which, (5.33), (9.7), we infer

$$\begin{aligned} \|\nabla \mathbf{p}'_1(Y; X)\|_{W^{N,3}} &\lesssim |\partial_3 Y|_0^{\frac{1}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\partial_3 X|_{N+1} + |\partial_3 Y|_0^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_{N+1} \\ (9.10) \quad &\quad + (|\partial_3 Y|_{N+1}^{\frac{1}{3}} \|\partial_3 Y\|_{N+1}^{\frac{2}{3}} + |\nabla Y|_{N+1} |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}}) |\partial_3 X|_1 \\ &\quad + (|\partial_3 Y|_{N+1} |\partial_3 Y|_0^{\frac{1}{3}} + |\nabla Y|_{N+1} |\partial_3 Y|_1^{\frac{4}{3}}) \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1. \end{aligned}$$

Together with (9.7), we deduce that

$$\begin{aligned} \|I_3\|_N &\lesssim |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}} (|\partial_3 X|_{N+1} \|\nabla W\|_1 + |\partial_3 X|_1 \|\nabla W\|_{N+1} + |\nabla Y|_{N+1} |\partial_3 X|_1 \|\nabla W\|_1) \\ &\quad + |\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} (|\nabla X|_{N+1} \|\nabla W\|_1 + |\nabla X|_1 \|\nabla W\|_{N+1} + |\nabla Y|_{N+1} |\nabla X|_1 \|\nabla W\|_1) \\ &\quad + |\partial_3 Y|_{N+1}^{\frac{1}{3}} \|\partial_3 Y\|_{N+1}^{\frac{2}{3}} |\partial_3 X|_1 \|\nabla W\|_1 + |\partial_3 Y|_{N+1} |\partial_3 Y|_0^{\frac{1}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} |\nabla X|_1 \|\nabla W\|_1. \end{aligned}$$

(iv) H^N estimate of I_4 . We first get, by applying Moser type inequality, that

$$\|I_4\|_N \lesssim \|\nabla \mathbf{p}''_1(Y; X, W)\|_N + |\nabla Y|_N \|\nabla \mathbf{p}''_1(Y; X, W)\|_0.$$

We first deal with the estimate of $\|\nabla \mathbf{p}_1''(Y; X, W)\|_0$. Indeed by (4.15), we have

$$\begin{aligned}
\|\nabla \mathbf{p}_1''(Y; X, W)\|_0 &\lesssim |\nabla Y|_0 \|\nabla \mathbf{p}_1''(Y; X, W)\|_0 + \|(\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)\|_{\dot{H}^1} \|\nabla \mathbf{p}_1\|_{L^3} \\
&\quad + |(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)|_0 \|\nabla \mathbf{p}_1'(Y; W)\|_0 + (1 + \|\mathcal{A}\|_{B_{2,1}^{\frac{3}{2}}}) \|\partial_3 X \otimes \partial_3 W\|_1 \\
&\quad + \|(\mathcal{A}\mathcal{A}^t - Id)'(Y; W)\|_1 \|\nabla \mathbf{p}_1'(Y; X)\|_{L^3} + |\partial_3 Y \otimes \partial_3 Y|_1 \|\mathcal{A}''(Y; X, W)\|_1 \\
&\quad + |\mathcal{A}'(Y; X)|_1 \|\partial_3 Y \otimes \partial_3 W\|_1 + \|\mathcal{A}'(Y; W)\|_1 \|\partial_3 Y \otimes \partial_3 X\|_1 \\
&\quad + |\mathcal{A}'(Y; X)|_0 (\|\partial_3 Y \otimes \partial_3 W\|_1 + \|\mathcal{A}'(Y; W)(\partial_3 Y \otimes \partial_3 Y)\|_1) \\
&\quad + \|\mathcal{A}'(Y; W)\|_0 |\mathcal{A}(\partial_3 Y \otimes \partial_3 X)|_1 + \|\mathcal{A}'(Y; X)(\partial_3 Y \otimes \partial_3 Y)\|_1 \\
&\quad + \|\mathcal{A}''(Y; X, W)\|_0 \|\partial_3 Y \otimes \partial_3 Y\|_1,
\end{aligned}$$

from which, (5.36) and (9.7), we deduce that

$$\begin{aligned}
(9.11) \quad \|\nabla \mathbf{p}_1''(Y; X, W)\|_0 &\lesssim |\partial_3 X|_1 (\|\partial_3 W\|_1 + (|\partial_3 Y|_1 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}}) \|\nabla W\|_1) \\
&\quad + |\nabla X|_1 (|\partial_3 Y|_1 \|\partial_3 W\|_1 + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) \|\nabla W\|_1).
\end{aligned}$$

In general, along the same line to the proof of (9.11), we get, by using the estimates (5.33), (5.36), (9.7), (9.8), (9.10) and (9.11), that

$$\begin{aligned}
(9.12) \quad \|\nabla \mathbf{p}_1''(Y; X, W)\|_N &\lesssim (|\partial_3 X|_{N+1} + |\partial_3 X|_0 |\nabla Y|_{N+1}) \|\partial_3 W\|_0 + |\partial_3 X|_0 \|\partial_3 W\|_{N+1} \\
&\quad + |\partial_3 Y|_0 (|\nabla X|_{N+1} \|\partial_3 W\|_0 + |\nabla X|_0 \|\partial_3 W\|_{N+1}) \\
&\quad + (|\partial_3 Y|_{N+1} + |\partial_3 Y|_1 |\nabla Y|_{N+1}) |\nabla X|_1 \|\partial_3 W\|_1 + (|\partial_3 Y|_0 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}}) \\
&\quad \times ((|\partial_3 X|_{N+1} + |\nabla X|_1 |\partial_3 Y|_{N+1}) \|\nabla W\|_1 + |\partial_3 X|_1 \|\nabla W\|_{N+1}) \\
&\quad + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) ((|\nabla X|_{N+1} + |\nabla X|_1 |\nabla Y|_{N+1}) \|\nabla W\|_1 + |\nabla X|_1 \|\nabla W\|_{N+1}) \\
&\quad + (|\partial_3 Y|_{N+1} + |\partial_3 Y|_{N+1}^{\frac{1}{3}} \|\partial_3 Y\|_{N+1}^{\frac{2}{3}} + |\nabla Y|_{N+1} (|\partial_3 Y|_1 + |\partial_3 Y|_0^{\frac{1}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}})) |\partial_3 X|_1 \|\nabla W\|_1.
\end{aligned}$$

The same estimate holds for $\|I_4\|_N$. By summing up the estimates of I_1, I_2, I_3 and I_4 , we achieve

$$\begin{aligned}
(9.13) \quad \|f_1''(Y; X, W)\|_N &\lesssim (|\partial_3 X|_{N+1} + |\partial_3 X|_0 |\nabla Y|_{N+1}) \|\partial_3 W\|_0 \\
&\quad + |\partial_3 Y|_0 (|\nabla X|_{N+1} \|\partial_3 W\|_0 + |\nabla X|_0 \|\partial_3 W\|_{N+1}) + |\partial_3 X|_0 \|\partial_3 W\|_{N+1} \\
&\quad + (|\partial_3 Y|_{N+1} + |\partial_3 Y|_1 |\nabla Y|_{N+1}) |\nabla X|_1 \|\partial_3 W\|_1 + (|\partial_3 Y|_0 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}}) \\
&\quad \times ((|\partial_3 X|_{N+1} + |\nabla X|_1 |\partial_3 Y|_{N+1}) \|\nabla W\|_1 + |\partial_3 X|_1 \|\nabla W\|_{N+1}) \\
&\quad + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) ((|\nabla X|_{N+1} + |\nabla X|_1 |\nabla Y|_{N+1}) \|\nabla W\|_1 + |\nabla X|_1 \|\nabla W\|_{N+1}) \\
&\quad + (|\partial_3 Y|_{N+1} + |\partial_3 Y|_{N+1}^{\frac{1}{3}} \|\partial_3 Y\|_{N+1}^{\frac{2}{3}} + (|\partial_3 Y|_1 + |\partial_3 Y|_0^{\frac{1}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}}) |\nabla Y|_{N+1}) |\partial_3 X|_1 \|\nabla W\|_1.
\end{aligned}$$

Then (9.2) follows from (9.9) and (9.13). Exactly along the same line, we can prove (9.3), and we omit the details here. This complete the proof of Proposition 9.1. \square

9.2. The estimate of $\|f''(Y; X, W)\|_{\delta, N}$.

Proposition 9.2. *Let f_m'' , $m = 0, 1, 2$ be given in (4.13) and (4.14), the norm $\|\cdot\|_{\delta, N}$ be given by (2.39). Then under the assumption of (2.41), we have*

$$\begin{aligned}
(9.14) \quad \|f_0''(Y; X, W)\|_{\delta, N} &\lesssim |Y_t|_1 (\|\nabla X\|_{N+6} \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_{N+6}) \\
&\quad + (\|\nabla Y_t\|_{N+6} + |Y_t|_1 \|\nabla Y\|_{N+6}) (\|\nabla X\|_0 \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_0) \\
&\quad + \|\nabla X\|_0 \|\nabla W_t\|_{N+6} + (\|\nabla X\|_{N+6} + \|\nabla X\|_0 \|\nabla Y\|_{N+6}) \|\nabla W_t\|_0 \\
&\quad + \|\nabla W\|_0 \|\nabla X_t\|_{N+6} + (\|\nabla W\|_{N+6} + \|\nabla W\|_0 \|\nabla Y\|_{N+6}) \|\nabla X_t\|_0,
\end{aligned}$$

and

$$(9.15) \quad \|f_1''(Y; X, W)\|_{\delta, N} \lesssim f_4(\partial_3 Y, \partial_3 X, \partial_3 W) \quad \text{and}$$

$$(9.16) \quad \|f_2''(Y; X, W)\|_{\delta, N} \lesssim f_4(Y_t, X_t, W_t),$$

where the functional $f_4(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ is given by

$$\begin{aligned} f_4(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \stackrel{\text{def}}{=} & (\|\mathfrak{z}\|_0 + |\mathfrak{x}|_0 \|\nabla W\|_0 + \|\mathfrak{x}\|_0 \|\nabla W\|_0) \|\mathfrak{y}\|_{N+6} + (\|\mathfrak{x}\|_0 \|\nabla X\|_0 + \|\mathfrak{y}\|_0) \|\mathfrak{z}\|_{N+6} \\ & + |\mathfrak{y}|_0 \|\mathfrak{z}\|_0 \|\nabla Y\|_{N+6} + |\mathfrak{x}|_0 (\|\mathfrak{z}\|_0 \|\nabla X\|_{N+6} + \|\nabla X\|_0 \|\mathfrak{z}\|_{N+6} + \|\mathfrak{y}\|_0 \|\nabla W\|_{N+6}) \\ & + (\|\mathfrak{x}\|_{N+6} + \|\mathfrak{x}\|_0 \|\nabla Y\|_{N+6}) (\|\nabla X\|_0 \|\mathfrak{z}\|_0 + \|\nabla X\|_1 \|\mathfrak{z}\|_1) \\ & + (\|\mathfrak{x}\|_{N+6} + \|\mathfrak{x}\|_3 \|\nabla Y\|_{N+6}) (|\mathfrak{y}|_1 \|\nabla W\|_1 + \|\mathfrak{y}\|_1 \|\nabla W\|_0) \\ & + |\mathfrak{x}|_1 \|\mathfrak{x}\|_3 (\|\nabla X\|_{N+6} (\|\nabla W\|_0 + \|\nabla W\|_0) + (\|\nabla X\|_0 + \|\nabla X\|_0) \|\nabla W\|_{N+6}) \\ & + (|\mathfrak{x}|_1 \|\mathfrak{x}\|_{N+6} + \|\mathfrak{x}\|_1 \|\mathfrak{x}\|_3 \|\nabla Y\|_{N+6}) (\|\nabla X\|_0 \|\nabla W\|_1 + \|\nabla X\|_1 \|\nabla W\|_0). \end{aligned}$$

Lemma 9.2. *Let $s > 0$. Then under the assumption of (2.41), one has*

$$(9.17) \quad \|\mathcal{A}'(Y; X)\|_{\dot{B}_{2,1}^s} + \|(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\|_{\dot{B}_{2,1}^s} \lesssim \|\nabla X\|_{\dot{B}_{2,1}^s} + \|\nabla X\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^s},$$

$$(9.18) \quad \|\mathcal{A}''(Y; X, W)\|_{\dot{B}_{1,1}^s} \lesssim (\|\nabla X\|_{\dot{B}_{2,1}^s} + \|\nabla X\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^s}) \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_{\dot{B}_{2,1}^s},$$

and

$$(9.19) \quad \begin{aligned} \|(\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)\|_{\dot{B}_{1,1}^s} & \lesssim (\|\nabla X\|_{\dot{B}_{2,1}^s} + \|\nabla X\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^s}) \|\nabla W\|_0 \\ & + (\|\nabla W\|_{\dot{B}_{2,1}^s} + \|\nabla W\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^s}) \|\nabla X\|_0. \end{aligned}$$

Proof. Indeed by virtue of (4.4), we get, by applying the law of product (2.48), that

$$\|\mathcal{A}'(Y; X)\|_{\dot{B}_{2,1}^s} \leq (1 + \|\mathcal{A} - Id\|_0) \|\nabla X\mathcal{A}\|_{\dot{B}_{2,1}^s} + \|\mathcal{A} - Id\|_{\dot{B}_{2,1}^s} \|\nabla X\mathcal{A}\|_0.$$

Similarly, we deduce estimates for $(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)$ from (4.5) and complete the proof of (9.17). While it follows from (4.11) and the law of product (5.1) that

$$\|\mathcal{A}''(Y; X, W)\|_{\dot{B}_{1,1}^s} \lesssim \|\nabla X\|_{\dot{B}_{2,1}^s} \|\mathcal{A}\nabla W\|_0 + \|\nabla X\|_0 \|\mathcal{A}\nabla W\|_{\dot{B}_{2,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla X\|_0 \|\nabla W\|_0,$$

which yields (9.18).

Finally it is easy to observe from the law of product and the previous estimates that

$$\|\mathcal{A}''(Y; X, W)\mathcal{A}^t\|_{\dot{B}_{1,1}^s} \lesssim (\|\nabla X\|_{\dot{B}_{2,1}^s} + \|\nabla X\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^s}) \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_{\dot{B}_{2,1}^s},$$

and

$$\begin{aligned} \|\mathcal{A}'(Y; X)(\mathcal{A}^t)'(Y; W)\|_{\dot{B}_{1,1}^s} & \lesssim (\|\nabla X\|_{\dot{B}_{2,1}^s} + \|\nabla X\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^s}) \|\nabla W\|_0 \\ & + (\|\nabla W\|_{\dot{B}_{2,1}^s} + \|\nabla W\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^s}) \|\nabla X\|_0, \end{aligned}$$

so that we can deduce (9.19) from (9.4). \square

Proof of Proposition 9.2. Again we divide the proof of this proposition into the following steps:

Step 1. Estimate of $f_0''(Y; X, W)$. We first deduce from the law of product (5.1) and Lemmas 9.1, 9.2 that

$$\begin{aligned} \|(\mathcal{A}\mathcal{A}^t - Id)''(Y; X, W)\nabla Y_t\|_{\dot{B}_{1,1}^{s+1}} & \lesssim |Y_t|_1 (\|\nabla W\|_0 \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla X\|_0 \|\nabla W\|_{\dot{B}_{2,1}^{s+1}}) \\ & + (\|\nabla Y_t\|_{\dot{B}_{2,1}^{s+1}} + |Y_t|_1 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) (\|\nabla X\|_0 \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_0), \end{aligned}$$

and

$$\|(\mathcal{A}\mathcal{A}^t - Id)'(Y; X)\nabla W_t\|_{\dot{B}_{1,1}^{s+1}} \lesssim \|\nabla X\|_0 \|\nabla W_t\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla X\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|\nabla W_t\|_0,$$

and

$$\|(\mathcal{A}\mathcal{A}^t - Id)'(Y; W)\nabla X_t\|_{\dot{B}_{1,1}^{s+1}} \lesssim \|\nabla W\|_0 \|\nabla X_t\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla W\|_{\dot{B}_{2,1}^{s+1}} + |\nabla W|_0 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|\nabla X_t\|_0.$$

Hence by virtue of (4.13), we conclude that

$$(9.20) \quad \begin{aligned} \|f_0''(Y; X, W)\|_{\dot{B}_{1,1}^s} &\lesssim |Y_t|_1 (\|\nabla W\|_0 \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + \|\nabla X\|_0 \|\nabla W\|_{\dot{B}_{2,1}^{s+1}}) \\ &\quad + (\|\nabla Y_t\|_{\dot{B}_{2,1}^{s+1}} + |Y_t|_1 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) (|\nabla X|_0 \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_0) \\ &\quad + \|\nabla X\|_0 \|\nabla W_t\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + |\nabla X|_0 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|\nabla W_t\|_0 \\ &\quad + \|\nabla W\|_0 \|\nabla X_t\|_{\dot{B}_{2,1}^{s+1}} + (\|\nabla W\|_{\dot{B}_{2,1}^{s+1}} + |\nabla W|_0 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|\nabla X_t\|_0. \end{aligned}$$

Then (9.14) follows from (9.20) and the interpolation inequality (5.22).

Step 2. Estimate of $f_m''(Y; X, W)$. Again we only present the estimates of $f_1''(Y; X, W)$. Recall (9.5), we shall split the estimate of $f_1''(Y; X, W)$ into the following 4 parts:

(i) Estimate of I_1 . It follows from (5.9) and (9.18) that

$$\begin{aligned} \|I_1\|_{\dot{B}_{1,1}^s} &\lesssim \|(\mathcal{A}^t)''(Y; X, W)\|_{\dot{B}_{1,1}^s} |\nabla \mathbf{p}_1|_0 + \|(\mathcal{A}^t)''(Y; X, W)\|_0 \|\nabla \mathbf{p}_1\|_{\dot{B}_{2,1}^s} \\ &\lesssim (\|\nabla X\|_{\dot{B}_{2,1}^s} \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_{\dot{B}_{2,1}^s}) |\partial_3 Y|_0 \|\partial_3 Y\|_3 \\ &\quad + (\|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + \|\partial_3 Y\|_3 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) |\partial_3 Y|_0 |\nabla X|_0 \|\nabla W\|_0. \end{aligned}$$

(ii) Estimate of I_2 . We deduce from (5.18) and (5.36) that

$$\begin{aligned} \|I_2\|_{\dot{B}_{1,1}^s} &\lesssim |(\mathcal{A}^t)'(Y; X)|_0 \|\nabla(\mathbf{p}_1'(Y; W))\|_{\dot{B}_{1,1}^s} + \|(\mathcal{A}^t)'(Y; X)\|_{\dot{B}_{2,1}^s} \|\nabla(\mathbf{p}_1'(Y; W))\|_0 \\ &\lesssim (|\partial_3 Y|_1 \|\partial_3 W\|_1 + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) \|\nabla W\|_1) \|\nabla X\|_{\dot{B}_{2,1}^s} \\ &\quad + \left((\|\partial_3 W\|_1 + |\partial_3 Y|_1 \|\nabla W\|_1) \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + (\|\partial_3 W\|_1 + \|\partial_3 Y\|_3 \|\nabla W\|_1) \times \right. \\ &\quad \left. \times |\partial_3 Y|_1 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + (\|\partial_3 W\|_{\dot{B}_{2,1}^{s+1}} + |\partial_3 Y|_0 \|\nabla W\|_{\dot{B}_{2,1}^{s+1}}) \|\partial_3 Y\|_0 \right) |\nabla X|_0. \end{aligned}$$

(iii) Estimate of I_3 . Similar to the estimate of I_2 , we have

$$\begin{aligned} \|I_3\|_{\dot{B}_{1,1}^s} &\lesssim (|\partial_3 Y|_1 \|\partial_3 X\|_1 + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1^2) \|\nabla X\|_1) \|\nabla W\|_{\dot{B}_{2,1}^s} \\ &\quad + \left((\|\partial_3 X\|_1 + |\partial_3 Y|_1 \|\nabla X\|_1) \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + (\|\partial_3 X\|_1 + \|\partial_3 Y\|_3 \|\nabla X\|_1) \times \right. \\ &\quad \left. \times |\partial_3 Y|_1 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + (|\partial_3 Y|_0 \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + |\nabla W|_0 \|\partial_3 X\|_{\dot{B}_{2,1}^{s+1}}) \|\partial_3 Y\|_0 \right) |\nabla W|_0. \end{aligned}$$

(iv) Estimate of I_4 . We first deduce from the law of product (5.1) that

$$\|I_4\|_{\dot{B}_{1,1}^s} \lesssim \|\nabla(\mathbf{p}_m''(Y; X, W))\|_{\dot{B}_{1,1}^s} + \|\nabla Y\|_{\dot{B}_{2,1}^s} \|\nabla(\mathbf{p}_m''(Y; X, W))\|_0.$$

Thanks to (9.11), it remains to handle the estimate of $\|\nabla(\mathbf{p}_m''(Y; X, W))\|_{\dot{B}_{1,1}^s}$. As a matter of fact, thanks to (4.15), by applying the laws of product, (2.48) and (5.1), and using the estimates (5.9),

(5.18), (5.36), (9.11) and (9.18), we obtain

$$\begin{aligned}
(9.21) \quad & \|\nabla p_1''(Y; X, W)\|_{\dot{B}_{1,1}^s} \lesssim (\|\partial_3 W\|_0 + |\partial_3 Y|_0 \|\nabla W\|_0 + \|\partial_3 Y\|_0 \|\nabla W\|_0) \|\partial_3 X\|_{\dot{B}_{2,1}^{s+1}} \\
& + (\|\nabla X\|_0 \|\partial_3 W\|_0 + \|\nabla X\|_1 \|\partial_3 W\|_1 + \|\partial_3 X\|_1 \|\nabla W\|_1 + \|\partial_3 X\|_1 \|\nabla W\|_0) \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} \\
& + (\|\partial_3 X\|_0 + \|\nabla X\|_0 \|\partial_3 Y\|_0 + \|\partial_3 Y\|_0 \|\nabla X\|_0) \|\partial_3 W\|_{\dot{B}_{2,1}^{s+1}} + (|\partial_3 X|_0 \|\partial_3 W\|_0 \\
& + |\partial_3 Y|_0 (\|\nabla X\|_0 \|\partial_3 W\|_0 + \|\nabla X\|_1 \|\partial_3 W\|_1) + |\partial_3 Y|_1 \|\partial_3 X\|_1 \|\nabla W\|_0 \\
& + (|\partial_3 Y|_0 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}}) |\partial_3 X|_1 \|\nabla W\|_1) \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + |\partial_3 Y|_0 \left((\|\partial_3 W\|_0 \right. \\
& + \|\partial_3 Y\|_0 \|\nabla W\|_0) \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} + (\|\partial_3 X\|_0 + \|\partial_3 Y\|_0 \|\nabla X\|_0) \|\nabla W\|_{\dot{B}_{2,1}^{s+1}} \Big) \\
& + (|\partial_3 Y|_1^{\frac{4}{3}} \|\partial_3 Y\|_0^{\frac{2}{3}} + |\partial_3 Y|_1 \|\partial_3 Y\|_3) (\|\nabla X\|_{\dot{B}_{2,1}^{s+1}} \|\nabla W\|_0 + \|\nabla X\|_0 \|\nabla W\|_{\dot{B}_{2,1}^{s+1}}) \\
& + (|\partial_3 Y|_1 \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + |\partial_3 Y|_1 \|\partial_3 Y\|_3 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) (\|\nabla X\|_0 \|\nabla W\|_1 + \|\nabla X\|_1 \|\nabla W\|_0),
\end{aligned}$$

The same estimate holds for $\|I_4\|_{\dot{B}_{1,1}^s}$.

By summing up the above estimates of I_1, I_2, I_3, I_4 , we arrive at

$$\begin{aligned}
(9.22) \quad & \|f_1''(Y; X, W)\|_{\dot{B}_{1,1}^s} \lesssim (\|\partial_3 W\|_0 + |\partial_3 Y|_0 \|\nabla W\|_0 + \|\partial_3 Y\|_0 \|\nabla W\|_0) \|\partial_3 X\|_{\dot{B}_{2,1}^{s+1}} \\
& + (\|\partial_3 X\|_0 + |\partial_3 Y|_0 \|\nabla X\|_0 + \|\partial_3 Y\|_0 \|\nabla X\|_0) \|\partial_3 W\|_{\dot{B}_{2,1}^{s+1}} \\
& + (|\partial_3 Y|_0 \|\partial_3 W\|_0 + (|\partial_3 Y|_1 \|\nabla W\|_0 + |\partial_3 Y|_0 \|\nabla W\|_0) \|\partial_3 Y\|_3) \|\nabla X\|_{\dot{B}_{2,1}^{s+1}} \\
& + (|\partial_3 Y|_0 \|\partial_3 X\|_0 + (|\partial_3 Y|_1 \|\nabla X\|_0 + |\partial_3 Y|_0 \|\nabla X\|_0) \|\partial_3 Y\|_3) \|\nabla W\|_{\dot{B}_{2,1}^{s+1}} \\
& + |\partial_3 X|_0 \|\partial_3 W\|_0 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}} + (\|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + |\partial_3 Y|_1 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) \|\partial_3 X\|_1 \|\nabla W\|_0 \\
& + (\|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + |\partial_3 Y|_0 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) (\|\nabla X\|_0 \|\partial_3 W\|_0 + \|\nabla X\|_1 \|\partial_3 W\|_1) \\
& + (\|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + (|\partial_3 Y|_0 + |\partial_3 Y|_1^{\frac{1}{3}} \|\partial_3 Y\|_1^{\frac{2}{3}}) \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) |\partial_3 X|_1 \|\nabla W\|_1 \\
& + (|\partial_3 Y|_1 \|\partial_3 Y\|_{\dot{B}_{2,1}^{s+1}} + |\partial_3 Y|_1 \|\partial_3 Y\|_3 \|\nabla Y\|_{\dot{B}_{2,1}^{s+1}}) (\|\nabla X\|_0 \|\nabla W\|_1 + \|\nabla X\|_1 \|\nabla W\|_0).
\end{aligned}$$

Then (9.15) follows from (9.22) and interpolation inequality (5.22). Finally, the proof of (9.16) follows along the same line to that of (9.15). We omit the details here. This completes the proof of Proposition 9.2. \square

10. THE PROOF OF THEOREM 2.1

The goal of this section is to prove Theorem 2.1 by using Nash-Moser scheme. The key ingredients are the uniform estimates of the approximate solutions obtained in Propositions 2.6, 2.7 and 2.8, which we will prove by induction in what follows.

10.1. The estimates of Y_0 . Recall that Y_0 solves the linear equation (2.66). Let $\bar{N}_0 = N_0 + 6$, for $\eta \in]0, 1[$, we choose the initial data $(Y^{(0)}, Y^{(1)})$ such that (2.20) holds for $L_0 = N_0 + 12$. Then we get, by applying (2.32) of Proposition 2.1, that

$$\begin{aligned}
(10.1) \quad & |\partial_3 Y_0|_{1, \bar{N}_0} + |\partial_t Y_0|_{\frac{3}{2} - \delta, \bar{N}_0} + |Y_0|_{\frac{1}{2}, \bar{N}_0} \\
& \leq C_{N_0} (\| |D|^{2\delta} (Y^{(0)}, Y^{(1)}) \|_{L^1} + \| |D|^{\bar{N}_0 + 4} (|D|^2 Y^{(0)}, Y^{(1)}) \|_{L^1}) \leq \eta.
\end{aligned}$$

Note that

$$\| |D|^{-1} h \|_0 \leq \left(\int_{|\xi| \leq 1} \frac{1}{|\xi|^2} |\hat{h}(\xi)|^2 d\xi \right)^{\frac{1}{2}} + \|h\|_0 \leq |\hat{h}|_0 + \|h\|_0 \leq \|h\|_{L^1} + \|h\|_0,$$

so that we get, by applying (2.33), (2.34) and (2.35) of Proposition 2.1, that

$$\begin{aligned}
(10.2) \quad & \| |D|^{-1}(\partial_3 Y_0, \partial_t Y_0) \|_{0, \bar{N}_0+2} + \|\nabla Y_0\|_{0, \bar{N}_0+1} + \|\nabla \partial_t Y_0\|_{1, \bar{N}_0-1} \\
& + \|(\partial_t Y_0, \partial_3 Y_0)\|_{\frac{1}{2}, \bar{N}_0+1} + \|\partial_t Y_0\|_{L_t^2(H^{\bar{N}_0+2})} + \|(\partial_3 Y_0, \langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_0)\|_{L_t^2(H^{\bar{N}_0+1})} \\
& \leq C_{\bar{N}_0} \| |D|^{-1}(\partial_3 Y^{(0)}, Y^{(1)}, \Delta Y^{(0)}) \|_{\bar{N}_0+2} \\
& \leq C_{\bar{N}_0} (\|(\partial_3 Y^{(0)}, Y^{(1)}, \Delta Y^{(0)})\|_{L^1} + \|(\partial_3 Y^{(0)}, Y^{(1)}, \Delta Y^{(0)})\|_{\bar{N}_0+1}) \leq \eta.
\end{aligned}$$

By virtue of (10.1) and (10.2), we deduce from Proposition 8.2 that

$$\begin{aligned}
(10.3) \quad & \| \langle t \rangle |D|^{-1} f(Y_0) \|_{L_t^2(H^{N_0+1})} \lesssim |\partial_3 Y_0|_{1,0} \|\partial_3 Y_0\|_{L_t^2(H^{N_0+1})} + |\partial_t Y_0|_{1,0} \|\partial_t Y_0\|_{L_t^2(H^{N_0+1})} \\
& + |\nabla Y_0|_{\frac{1}{2},0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_0\|_{L_t^2(H^{N_0+1})} + |\nabla Y_0|_{\frac{1}{2}, N_0+1} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_0\|_{L_t^2(L^2)} \\
& + |\nabla Y_0|_{0, N_0+1} (|\partial_3 Y_0|_{1,0} \|\partial_3 Y_0\|_{L_t^2(H^1)} + |\partial_t Y_0|_{1,0} \|\partial_t Y_0\|_{L_t^2(H^1)}) \lesssim C_{N_0} \eta^2,
\end{aligned}$$

and

$$\begin{aligned}
(10.4) \quad & \| |D|^{-1} f(Y_0) \|_{\frac{3}{2}, N_0+1} \lesssim |\partial_3 Y_0|_{1,0} \|\partial_3 Y_0\|_{\frac{1}{2}, N_0+1} + |\partial_t Y_0|_{1,0} \|\partial_t Y_0\|_{\frac{1}{2}, N_0+1} \\
& + |\nabla Y_0|_{\frac{1}{2},0} \|\nabla \partial_t Y_0\|_{1, N_0+1} + |\nabla Y_0|_{\frac{1}{2}, N_0+1} \|\nabla \partial_t Y_0\|_{1,0} \\
& + |\nabla Y_0|_{0, N_0+1} (|\partial_3 Y_0|_{1,0} \|\partial_3 Y_0\|_{\frac{1}{2},1} + |\partial_t Y_0|_{1,0} \|\partial_t Y_0\|_{\frac{1}{2},1}) \lesssim C_{N_0} \eta^2.
\end{aligned}$$

Similarly, we deduce from Proposition 8.1 and (10.1), (10.2) that

$$\begin{aligned}
(10.5) \quad & \| \langle t \rangle^{\frac{1}{2}} f(Y_0) \|_{L_t^2(\delta, N_0)} \lesssim \|\nabla Y_0\|_{0,0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_0\|_{L_t^2(H^{N_0+6})} + \|\nabla Y_0\|_{0, N_0+6} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_0\|_{L_t^2(L^2)} \\
& + \|\partial_3 Y_0\|_{\frac{1}{2},0} \|\partial_3 Y_0\|_{L_t^2(H^{N_0+6})} + \|\partial_t Y_0\|_{\frac{1}{2},0} \|\partial_t Y_0\|_{L_t^2(H^{N_0+6})} \\
& + \|\nabla Y_0\|_{0, N_0+6} (|\partial_3 Y_0|_{\frac{1}{2},0} \|\partial_3 Y_0\|_{L_t^2(H^1)} + |\partial_t Y_0|_{\frac{1}{2},0} \|\partial_t Y_0\|_{L_t^2(H^1)}) \lesssim C_{N_0} \eta^2.
\end{aligned}$$

10.2. The proof of Proposition 2.7 and Proposition 2.8 from Proposition 2.6. Let us assume that

$$(10.6) \quad (P1, j), (P2, j), (P3, j) \text{ of Proposition 2.6 hold for } j \leq p,$$

we are going to prove Proposition 2.7 and Proposition 2.8.

Proof of Proposition 2.7. Notice from (2.69) that

$$|\partial_3 Y_{p+1}|_{k,N} \leq |\partial_3 Y_0|_{k,N} + \sum_{j=0}^p |\partial_3 X_j|_{k,N},$$

which together with (10.1) and (P2, j) with $j \leq p$ ensures that for $\frac{1}{2} \leq k \leq 1$, $0 \leq N \leq N_0$,

$$\begin{aligned}
(10.7) \quad & |\partial_3 Y_{p+1}|_{k,N} \leq C \eta \theta_{p+1}^{k-\frac{1}{2}-\gamma+\varepsilon N}, \quad \text{if } k - \frac{1}{2} - \gamma + \varepsilon N \geq \bar{\varepsilon}, \\
& |\partial_3 Y_{p+1}|_{k,N} \leq C \eta, \quad \text{if } k - \frac{1}{2} - \gamma + \varepsilon N \leq -\bar{\varepsilon}.
\end{aligned}$$

While for $\hat{k} \stackrel{\text{def}}{=} \min(k, 1)$, $\hat{N} \stackrel{\text{def}}{=} \min(N, N_0)$, we observe from the property (S I) of smoothing operator S_{p+1} that

$$\begin{aligned}
& |S_{p+1} \partial_3 Y_{p+1}|_{k,N} \leq C |\partial_3 Y_{p+1}|_{k,N} \quad \text{for } \frac{1}{2} \leq k \leq 1, 0 \leq N \leq N_0, \\
& |S_{p+1} \partial_3 Y_{p+1}|_{k,N} \leq C_{k,N} \theta_{p+1}^{\max(0, k-\hat{k})} \theta_{p+1}^{\varepsilon \max(0, N-\hat{N})} |\partial_3 Y_{p+1}|_{\hat{k}, \hat{N}} \quad \text{for } k \geq 1 \text{ or } N \geq N_0,
\end{aligned}$$

the first inequalities of (I)(i) and (II)(i) of Proposition 2.7 then follow from (10.7).

Along the same line to proof of (10.7), we have

- for $1 - \delta \leq k \leq \frac{3}{2} - \delta$, $0 \leq N \leq N_0$,

$$(10.8) \quad \begin{aligned} |\partial_t Y_{p+1}|_{k,N} &\leq C\eta \theta_{p+1}^{k-(1-\delta)-\gamma+\varepsilon N}, & \text{if } k - (1-\delta) - \gamma + \varepsilon N \geq \bar{\varepsilon}, \\ |\partial_t Y_{p+1}|_{k,N} &\leq C\eta, & \text{if } k - (1-\delta) - \gamma + \varepsilon N \leq -\bar{\varepsilon}; \end{aligned}$$

- for $0 \leq k \leq \frac{1}{2}$, $0 \leq N \leq N_0$,

$$(10.9) \quad \begin{aligned} |Y_{p+1}|_{k,N} &\leq C\eta \theta_{p+1}^{k-\gamma+\varepsilon N}, & \text{if } k - \gamma + \varepsilon N \geq \bar{\varepsilon}, \\ |Y_{p+1}|_{k,N} &\leq C\eta, & \text{if } k - \gamma + \varepsilon N \leq -\bar{\varepsilon}. \end{aligned}$$

Then other inequalities in (I)(i) and (II)(i) of Proposition 2.7 follows.

(I)(ii) and (II)(ii) of Proposition 2.7 follow from property (S I) of the mollifying operator and the following fact

$$(10.10) \quad \begin{aligned} &\| |D|^{-1}(\partial_3 Y_{p+1}, \partial_t Y_{p+1}) \|_{0,N+2} + \|\nabla Y_{p+1}\|_{0,N+1} + \|\partial_t Y_{p+1}\|_{L_t^2(H^{N+2})} \\ &+ \|(\partial_t Y_{p+1}, \partial_3 Y_{p+1})\|_{\frac{1}{2},N+1} + \|\nabla \partial_t Y_{p+1}\|_{1,N-1} + \|(\partial_3 Y_{p+1}, \langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_{p+1})\|_{L_t^2(H^{N+1})} \\ &\leq \begin{cases} C\eta \theta_{p+1}^{-\beta+\varepsilon N}, & \text{for } -\beta + \varepsilon N \geq \bar{\varepsilon}, \quad N \leq N_0, \\ C\eta, & \text{for } -\beta + \varepsilon N \leq -\bar{\varepsilon}, \quad N \leq N_0, \end{cases} \end{aligned}$$

which is a direct consequence of (P1,j) of Proposition 2.6 for $j \leq p$ and (10.2).

Finally let us prove (III) of Proposition 2.7. Indeed it follows from property (S II) of S_{p+1} that

$$|(1 - S_{p+1})\partial_3 Y_{p+1}|_{\frac{1}{2},0} \leq C(\theta_{p+1}^{-\frac{1}{2}} |\partial_3 Y_{p+1}|_{1,0} + \theta_{p+1}^{-\varepsilon N_0} |\partial_3 Y_{p+1}|_{\frac{1}{2},N_0})$$

While due to (2.77) and (2.78), there hold $\frac{1}{2} - \gamma \geq \bar{\varepsilon}$ and $-\gamma + \varepsilon N_0 \geq \bar{\varepsilon}$, so that we can apply (10.7) to deduce that

$$(10.11) \quad |(1 - S_{p+1})\partial_3 Y_{p+1}|_{\frac{1}{2},0} \leq C\eta(\theta_{p+1}^{-\frac{1}{2}} \theta_{p+1}^{\frac{1}{2}-\gamma} + \theta_{p+1}^{-\varepsilon N_0} \theta_{p+1}^{-\gamma+\varepsilon N_0}) \leq C\eta \theta_{p+1}^{-\gamma}.$$

Using (10.7) once again gives rise to

$$(10.12) \quad |(1 - S_{p+1})\partial_3 Y_{p+1}|_{1,N} \leq C|\partial_3 Y_{p+1}|_{1,N} \leq \eta \theta_{p+1}^{\frac{1}{2}-\gamma+\varepsilon N} \quad \text{for } 0 \leq N \leq N_0,$$

$$(10.13) \quad |(1 - S_{p+1})\partial_3 Y_{p+1}|_{k,N_0} \leq C|\partial_3 Y_{p+1}|_{k,N_0} \leq \eta \theta_{p+1}^{k-\frac{1}{2}-\gamma+\varepsilon N_0} \quad \text{for } \frac{1}{2} \leq k \leq 1.$$

Interpolating between (10.11), (10.12) and (10.13) leads to

$$|(1 - S_{p+1})\partial_3 Y_{p+1}|_{k,N} \leq C\eta \theta_{p+1}^{k-\frac{1}{2}-\gamma+\varepsilon N}, \quad \text{for all } \frac{1}{2} \leq k \leq 1, \quad 0 \leq N \leq N_0.$$

The other two inequalities in (III) of Proposition 2.7 can be proved by the same procedure. This completes the proof of Proposition 2.7. \square

Let us now turn to the proof of Proposition 2.8.

Proof of Proposition 2.8. We shall divide the proof of this proposition by the following steps:

Step 1. The Proof of (IV) of Proposition 2.8. The proof of (IV) will be based on the following lemmas:

Lemma 10.1. *Let $e'_{p,j}, e''_{p,j}$, for $j = 0, 1, 2$, be given by (2.75). Then under the assumption of (10.6), one has*

$$(10.14) \quad \|\langle t \rangle^{\frac{1}{2}+k} |D|^{-1} (e''_{p,1} + e''_{p,2})\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \leq k \leq \frac{1}{2}, 0 \leq N \leq N_0 - 1;$$

$$(10.15) \quad \|\langle t \rangle^{\frac{1}{2}+k} |D|^{-1} (e'_{p,1} + e'_{p,2})\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \leq k \leq \frac{1}{2}, 0 \leq N \leq N_0 - 1;$$

$$(10.16) \quad \|\langle t \rangle^{\frac{1}{2}+k} |D|^{-1} e''_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} \quad \text{if } 0 \leq k \leq \alpha, 0 \leq N \leq N_0 - 2;$$

$$(10.17) \quad \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} e'_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} \quad \text{if } 0 \leq k \leq \alpha, 0 \leq N \leq N_0 - 2.$$

Lemma 10.2. *Under the assumption of Lemma 10.1, one has*

$$(10.18) \quad \| |D|^{-1} (e''_{p,1} + e''_{p,2}) \|_{1+k, N+1} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \leq k \leq \frac{1}{2}, 0 \leq N \leq N_0 - 1;$$

$$(10.19) \quad \| |D|^{-1} (e'_{p,1} + e'_{p,2}) \|_{1+k, N+1} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)} \quad \text{if } 0 \leq k \leq \frac{1}{2}, 0 \leq N \leq N_0 - 1;$$

$$(10.20) \quad \| |D|^{-1} e''_{p,0} \|_{1+k, N+1} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)} \quad \text{if } 0 \leq k \leq \frac{1}{2} - \delta, N \leq N_0 - 2;$$

$$(10.21) \quad \| |D|^{-1} e'_{p,0} \|_{1+k, N+1} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)} \quad \text{if } 0 \leq k \leq \frac{1}{2} - \delta, N \leq N_0 - 2.$$

Lemma 10.3. *Under the assumption of Lemma 10.1, for $0 \leq N \leq N_0 - 6$, there hold*

$$(10.22) \quad \| (e''_{p,1} + e''_{p,2}) \|_{L_t^1(\delta, N)} \lesssim \eta^2 \theta_p^{-\beta-\gamma+\bar{\varepsilon}(N+5)};$$

$$(10.23) \quad \| (e'_{p,1} + e'_{p,2}) \|_{L_t^1(\delta, N)} \lesssim \eta^2 \theta_p^{-\gamma+\bar{\varepsilon}(N+5)};$$

$$(10.24) \quad \| \langle t \rangle^{\frac{1}{2}} e''_{p,0} \|_{L_t^2(\delta, N)} \lesssim \eta^2 \theta_p^{-\beta-\gamma+\bar{\varepsilon}(N+5)};$$

$$(10.25) \quad \| \langle t \rangle^{\frac{1}{2}} e'_{p,0} \|_{L_t^2(\delta, N)} \lesssim \eta^2 \theta_p^{-\gamma+\bar{\varepsilon}(N+5)}.$$

We shall postpone the proof of the above lemmas in the Appendix A. It is easy to observe that (IV) (i) follows from Lemma 10.1, (IV) (ii) from Lemma 10.2, and (IV) (iii) from Lemma 10.3.

Step 2. The proof of (V) of Proposition 2.8. Recall (2.74) that

$$g_{p+1} = -(S_{p+1} - S_p)E_p - S_{p+1}e_p + (S_{p+1} - S_p)f(Y_0).$$

In the sequel, we shall handle term by term above.

•Estimates of $S_{p+1}e_p$

It follows from (IV) of Proposition 2.8 and property (S I) that for $k \geq 0$ and $N \geq 0$

$$(10.26) \quad \begin{aligned} & \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} S_{p+1}e_p\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}; \\ & \| |D|^{-1} S_{p+1}e_p \|_{1+k, N+1} \lesssim \eta^2 \theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}; \\ & \| \langle t \rangle^{\frac{1}{2}} S_{p+1}e_p \|_{L_t^2(\delta, N)} \lesssim \eta^2 \theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+5)}. \end{aligned}$$

Notice that the operator S_{p+1} contains a cutoff in the variable t of size θ_{p+1} so that

$$(10.27) \quad \| S_{p+1}e_p \|_{L_t^1(\delta, N)} \lesssim (\log \theta_{p+1})^{\frac{1}{2}} \| \langle t \rangle^{\frac{1}{2}} S_{p+1}e_p \|_{L_t^2(\delta, N)} \lesssim \eta^2 \theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+6)}.$$

•Estimates for $(S_{p+1} - S_p)E_p$

We first deduce from (IV) (i) of Proposition 2.8 that for $0 \leq k \leq \alpha$ and $0 \leq N \leq N_0 - 2$

$$(10.28) \quad \begin{aligned} \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} E_p\|_{L_t^2(H^{N+1})} &\leq \sum_{j=0}^{p-1} \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} e_j\|_{L_t^2(H^{N+1})} \\ &\lesssim \begin{cases} C\eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)} & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3) \geq \bar{\varepsilon}; \\ C\eta^2, & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3) \leq -\bar{\varepsilon}. \end{cases} \end{aligned}$$

In particular, due to the choice of parameters (2.77), (2.78), there hold

$$(10.29) \quad \frac{1}{2} - \gamma - \beta + 2\bar{\varepsilon} \geq \bar{\varepsilon}, \quad -\gamma - \beta + \bar{\varepsilon}(N_0 + 1) \geq \bar{\varepsilon},$$

we deduce from (10.28) and the property (S II) of $1 - S_p$ that

$$(10.30) \quad \begin{aligned} &\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} (S_{p+1} - S_p) E_p\|_{L_t^2(H^1)} \\ &\lesssim \theta_p^{-\alpha} \|\langle t \rangle^{\frac{1}{2}+\alpha} |D|^{-1} E_p\|_{L_t^2(H^1)} + \theta_p^{-\bar{\varepsilon}(N_0-1)} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1} E_p\|_{L_t^2(H^{N_0-1})} \\ &\lesssim \eta^2 (\theta_p^{-\alpha} \theta_p^{\alpha+\delta-\gamma-\beta+3\bar{\varepsilon}} + \theta_p^{-\bar{\varepsilon}(N_0-1)} \theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N_0+1)}) \lesssim \eta^2 \theta_{p+1}^{\delta-\gamma-\beta+3\bar{\varepsilon}}. \end{aligned}$$

On the other hand, for $k \leq \alpha$, $N \leq N_0 - 2$ with $k + \delta - \gamma - \beta + \bar{\varepsilon}(N + 3) \geq \bar{\varepsilon}$, we have

$$(10.31) \quad \begin{aligned} \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} (S_{p+1} - S_p) E_p\|_{L_t^2(H^{N+1})} &\lesssim \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} E_p\|_{L_t^2(H^{N+1})} \\ &\lesssim \eta^2 \theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}. \end{aligned}$$

Interpolating between (10.30) and (10.31), we conclude that

$$(10.32) \quad \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1} (S_{p+1} - S_p) E_p\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}$$

for $0 \leq k \leq \alpha$ and $0 \leq N \leq N_0 - 2$. This together with property (S I) of S_p ensures that (10.32) holds for any $k \geq 0$, $N \geq 0$.

Similarly we infer from (IV) (ii) of Proposition 2.8 that for $0 \leq k \leq \frac{1}{2} - \delta$, $0 \leq N \leq N_0 - 2$,

$$(10.33) \quad \begin{aligned} \||D|^{-1} E_p\|_{1+k, N+1} &\leq \sum_{j=0}^{p-1} \||D|^{-1} e_j\|_{1+k, N+1} \\ &\lesssim \begin{cases} C\eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}, & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2) \geq \bar{\varepsilon}; \\ C\eta^2, & \text{if } k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2) \leq -\bar{\varepsilon}. \end{cases} \end{aligned}$$

Then due to (10.29), we deduce from (10.33) and the property (S II) of $1 - S_p$ that

$$(10.34) \quad \begin{aligned} \||D|^{-1} (S_{p+1} - S_p) E_p\|_{1,1} &\lesssim \theta_p^{-\frac{1}{2}+\delta} \||D|^{-1} E_p\|_{\frac{3}{2}-\delta,1} + \theta_p^{-\bar{\varepsilon}(N_0-1)} \||D|^{-1} E_p\|_{1, N_0-1} \\ &\lesssim \eta^2 (\theta_p^{-\frac{1}{2}+\delta} \theta_p^{\frac{1}{2}-\gamma-\beta+2\bar{\varepsilon}} + \theta_p^{-\bar{\varepsilon}(N_0-1)} \theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N_0+1)}) \\ &\lesssim \eta^2 \theta_{p+1}^{\delta-\gamma-\beta+2\bar{\varepsilon}}, \end{aligned}$$

On the other hand, for $k \leq \frac{1}{2} - \delta$, $N \leq N_0 - 2$ such that $k + \delta - \gamma - \beta + \bar{\varepsilon}(N + 2) \geq \bar{\varepsilon}$, we get

$$(10.35) \quad \||D|^{-1} (S_{p+1} - S_p) E_p\|_{1+k, N+1} \lesssim \||D|^{-1} E_p\|_{1+k, N+1} \lesssim \eta^2 \theta_{p+1}^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}.$$

Interpolating between the inequalities (10.34) and (10.35), we achieve (10.35) for any $0 \leq k \leq \frac{1}{2} - \delta$, $0 \leq N \leq N_0 - 2$. This together with the property (S I) of S_p ensures that (10.35) holds for any $k \geq 0$ and $N \geq 0$.

It follows from (IV) (iii) of Proposition 2.8 that for $N \leq N_0 - 6$,

$$\begin{aligned} \|\langle t \rangle^{\frac{1}{2}} E_p\|_{L_t^2(\delta, N)} &\leq \sum_{j=0}^{p-1} \|\langle t \rangle^{\frac{1}{2}} e_j\|_{L_t^2(\delta, N)} \lesssim \eta^2 \sum_{j=0}^{p-1} \theta_j^{-\gamma + \bar{\varepsilon}(N+5)} \\ &\lesssim \begin{cases} \eta^2 \theta_{p+1}^{-\gamma + \bar{\varepsilon}(N+5)} & \text{if } -\gamma + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}; \\ \eta^2, & \text{if } -\gamma + \bar{\varepsilon}(N+5) \leq -\bar{\varepsilon}, \end{cases} \end{aligned}$$

which together with the property (S I) and compact support of mollifying operator ensures that for any $N \geq 0$

$$(10.36) \quad \|(S_{p+1} - S_p)E_p\|_{L_t^1(\delta, N)} \lesssim \begin{cases} \eta^2 \theta_{p+1}^{-\gamma + \bar{\varepsilon}(N+6)}, & \text{if } -\gamma + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}; \\ \eta^2 \theta_{p+1}^{\bar{\varepsilon}}, & \text{if } -\gamma + \bar{\varepsilon}(N+5) \leq -\bar{\varepsilon}. \end{cases}$$

• **Estimates for $(S_{p+1} - S_p)f(Y_0)$**

Recall (10.29), we get, by applying (S II) and (10.3), that

$$\begin{aligned} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1}(S_{p+1} - S_p)f(Y_0)\|_{L_t^2(H^1)} &\lesssim \theta_{p+1}^{-\frac{1}{2}} \|\langle t \rangle |D|^{-1}(S_{p+1} - S_p)f(Y_0)\|_{L_t^2(H^1)} \\ &\quad + \theta_{p+1}^{-\bar{\varepsilon}N_0} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1}(S_{p+1} - S_p)f(Y_0)\|_{L_t^2(H^{N_0+1})} \\ &\lesssim \eta^2 (\theta_{p+1}^{-\frac{1}{2}} + \theta_{p+1}^{-\bar{\varepsilon}N_0}) \lesssim \eta^2 \theta_{p+1}^{-\gamma - \beta + \bar{\varepsilon}}. \end{aligned}$$

Whereas for $k \leq \frac{1}{2}$ and $N \leq N_0$ with $k - \gamma - \beta + \bar{\varepsilon}(N+3) \geq \bar{\varepsilon}$, we deduce from (10.3) that

$$\begin{aligned} \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1}(S_{p+1} - S_p)f(Y_0)\|_{L_t^2(H^{N+1})} &\lesssim \|\langle t \rangle |D|^{-1}f(Y_0)\|_{L_t^2(H^{N_0+1})} \\ &\lesssim \eta^2 \leq \eta^2 \theta_{p+1}^{k-\gamma-\beta+\bar{\varepsilon}(N+3)}. \end{aligned}$$

Interpolating the above two inequalities gives rise to

$$\|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1}(S_{p+1} - S_p)f(Y_0)\|_{L_t^2(H^{N+1})} \leq \eta^2 \theta_{p+1}^{k-\gamma-\beta+\bar{\varepsilon}(N+3)}$$

for all $0 \leq k \leq \frac{1}{2}, 0 \leq N \leq N_0$. This together with the property (S I) of S_{p+1} ensures that

$$(10.37) \quad \|\langle t \rangle^{k+\frac{1}{2}} |D|^{-1}(S_{p+1} - S_p)f(Y_0)\|_{L_t^2(H^{N+1})} \leq \eta^2 \theta_{p+1}^{k-\gamma-\beta+\bar{\varepsilon}(N+3)}$$

for all $k \geq 0$ and $N \geq 0$.

Along the same line, it follows from (10.4) that for $k \geq 0, N \geq 0$,

$$(10.38) \quad \||D|^{-1}(S_{p+1} - S_p)f(Y_0)\|_{1+k, N+1} \leq \eta^2 \theta_{p+1}^{k-\gamma-\beta+\bar{\varepsilon}(N+3)}.$$

And it follows from (10.5) that if $-\gamma + \bar{\varepsilon}(N+5) \leq -\bar{\varepsilon}$ (implying $N \leq N_0$),

$$\|(S_{p+1} - S_p)f(Y_0)\|_{L_t^1(\delta, N)} \lesssim (\log \theta_{p+1})^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} f(Y_0)\|_{L_t^2(\delta, N_0)} \lesssim \eta^2 \theta_{p+1}^{\bar{\varepsilon}};$$

and if $-\gamma + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}$, one has

$$\begin{aligned} \|(S_{p+1} - S_p)f(Y_0)\|_{L_t^1(\delta, N)} &\lesssim (\log \theta_{p+1})^{\frac{1}{2}} \theta_{p+1}^{\bar{\varepsilon} \max(N-N_0, 0)} \|\langle t \rangle^{\frac{1}{2}} f(Y_0)\|_{L_t^2(\delta, N_0)} \\ &\lesssim \eta^2 \theta_{p+1}^{-\gamma + \bar{\varepsilon}(N+6)}, \end{aligned}$$

by using (S I) and the fact that $\bar{\varepsilon}(N_0 + 5) \geq \gamma$. Along with (10.26), (10.27), (10.32), (10.35), (10.36), (10.37), (10.38), we complete the proof of (V).

Step 3. The proof of (VI) of Proposition 2.8.

In the case when $-\gamma + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}$, we deduce from (V)(i), (V)(ii), (V)(iii) of Proposition 2.8 that

$$\begin{aligned} R_{N,\theta_{p+1}}(g_{p+1}) &= \|g_{p+1}\|_{L_t^1(\delta,N)} + \theta_{p+1}^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1} g_{p+1}\|_{L_t^2(H^{N+3})} + \log\langle \theta_{p+1} \rangle \| |D|^{-1} g_{p+1} \|_{\frac{3}{2}-\delta,N+3} \\ &\lesssim \eta^2 (\theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+6)} + \theta_{p+1}^{\frac{1}{2}+\delta-\gamma-\beta+\bar{\varepsilon}(N+5)} + \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+5)}) \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N}, \end{aligned}$$

provided that

$$(10.39) \quad 6\bar{\varepsilon} \leq \frac{1}{2}, \quad \beta \geq \delta + 5\bar{\varepsilon},$$

which are satisfied due to (2.77) and (2.76).

On the other hand, since $-\gamma + 6\bar{\varepsilon} \leq -\bar{\varepsilon}$, we deduce from (V)(i), (V)(ii) and (V)(iv) of Proposition 2.8 that

$$\begin{aligned} R_{0,\theta_{p+1}}(g_{p+1}) &= \|g_{p+1}\|_{L_t^1(\delta,0)} + \theta_{p+1}^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1} g_{p+1}\|_{L_t^2(H^3)} + \log\langle \theta_{p+1} \rangle \| |D|^{-1} g_{p+1} \|_{\frac{3}{2}-\delta,3} \\ &\lesssim \eta^2 (\theta_{p+1}^{\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}+\delta-\gamma-\beta+5\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+5\bar{\varepsilon}}) \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma}, \end{aligned}$$

due to (10.39) and $\frac{1}{2} - \gamma \geq \bar{\varepsilon}$. This finishes the proof of (VI) of Proposition 2.8 and hence the whole Proposition 2.8. \square

10.3. The proof of Proposition 2.6 from Proposition 2.7 and Proposition 2.8. Let us assume in this subsection that

$$(10.40) \quad \text{both Proposition 2.7 and Proposition 2.8 are valid}$$

we are going to prove (P1, $p+1$), (P2, $p+1$), (P3, $p+1$), that is, Proposition 2.6 is valid for $p+1$.

Proof of Proposition 2.6. We shall divide its proof into the following steps:

Step 1. The proof of (P3, $p+1$) of Proposition 2.6.

(P3, $p+1$) is a direct consequence of (10.7), (10.8), (10.9), (10.10) and the choices of parameters (see (2.77) and (2.76))

$$\beta \geq 3\bar{\varepsilon}, \quad C\eta \leq \delta_1, \quad \gamma \geq \delta + \varepsilon + 3\bar{\varepsilon}.$$

Step 2. The proof of (P1, $p+1$) of Proposition 2.6.

Recall that $X = X_{p+1}$ solves

$$(10.41) \quad X_{tt} - \Delta X_t - \partial_3^2 X = f'(S_{p+1}Y_{p+1}; X) + g_{p+1}.$$

Due to (P3, $p+1$), the hypotheses of Theorem 2.3 and (2.58) are satisfied, so that we can apply the energy estimate (2.59) to the system (10.41). When $N \geq 0$ with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$ and $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$, we deduce from (I) (i), (ii) of Proposition 2.7 that

$$\begin{aligned} \tilde{\gamma}_{\varepsilon,N+1}(S_{p+1}Y_{p+1}) &\lesssim |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\varepsilon,N+1} + |S_{p+1}\partial_t Y_{p+1}|_{1+\varepsilon,N+2} + |S_{p+1}\nabla Y_{p+1}|_{0,N+1} \\ &\quad + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+1} + 1 \lesssim \theta_{p+1}^{-\gamma+\varepsilon+\delta+\bar{\varepsilon}(N+2)} + \theta_{p+1}^{-\beta+\bar{\varepsilon}N}. \end{aligned}$$

Then in this case, we get, by applying the energy estimate (2.59) to the system (10.41) and using (V) (i), (V) (ii) of Proposition 2.8, that

$$\begin{aligned} &\| |D|^{-1}(\partial_3 X_{p+1}, \partial_t X_{p+1}) \|_{0,N+2} + \|\nabla X_{p+1}\|_{0,N+1} + \|(\partial_t X_{p+1}, \partial_3 X_{p+1})\|_{\frac{1}{2},N+1} \\ &\quad + \|\partial_t X_{p+1}\|_{L_t^2(H^{N+2})} + \|(\partial_3 X_{p+1}, \langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_{p+1})\|_{L_t^2(H^{N+1})} + \|\nabla \partial_t X_{p+1}\|_{1,N-1} \\ (10.42) \quad &\leq C_{\varepsilon,N} \left(\| |D|^{-1} g_{p+1} \|_{1+\varepsilon,N+1} + \|\langle t \rangle^{\frac{1+\varepsilon}{2}} g_{p+1} \|_{L_t^2(H^N)} \right. \\ &\quad \left. + \tilde{\gamma}_{\varepsilon,N+1}(S_{p+1}Y_{p+1}) (\| |D|^{-1} g \|_{1+\varepsilon,2} + \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g_{p+1} \|_{L_t^2(H^1)}) \right) \\ &\lesssim \eta^2 \theta_{p+1}^{\delta-\gamma-\beta+\bar{\varepsilon}N} (\theta_{p+1}^{\varepsilon+2\bar{\varepsilon}} + \theta_{p+1}^{\frac{\varepsilon}{2}+3\bar{\varepsilon}} + (\theta_{p+1}^{-\gamma+\varepsilon+\delta+2\bar{\varepsilon}} + \theta_{p+1}^{-\beta}) \theta_{p+1}^{\varepsilon+3\bar{\varepsilon}}) \lesssim \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}N} \end{aligned}$$

provided that $\gamma \geq \delta + \varepsilon + 3\bar{\varepsilon}$ which is satisfied due to (2.77), (2.76). Along the same line, we have

$$(10.43) \quad \begin{aligned} & \| |D|^{-1}(\partial_t X_{p+1}, \partial_3 X_{p+1}) \|_{0,2} + \|\nabla X_{p+1}\|_{0,1} + \|(\partial_t X_{p+1}, \partial_3 X_{p+1})\|_{\frac{1}{2},1} + \|\partial_t X_{p+1}\|_{L_t^2(H^2)} \\ & + \|(\partial_3 X_{p+1}, \langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_{p+1})\|_{L_t^2(H^1)} \leq C_\varepsilon \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g_{p+1}\|_{L_t^2(H^1)} \lesssim \eta \theta_{p+1}^{-\beta} \quad \text{and} \\ & \|\nabla \partial_t X_{p+1}\|_{1,0} \leq C_\varepsilon (\| |D|^{-1} g_{p+1}\|_{1+\varepsilon,2} + \|\langle t \rangle^{\frac{1+\varepsilon}{2}} |D|^{-1} g_{p+1}\|_{L_t^2(H^2)}) \leq \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}}. \end{aligned}$$

By interpolating the inequalities (10.42) and (10.43), we achieve (P1, $p+1$) for $N \geq 0$.

Step 3. The proof of (P2, $p+1$) of Proposition 2.6.

Notice that by definition $S_{p+1}Y_{p+1} = 0$ and $g_{p+1} = 0$ for $t \geq \theta_{p+1}$. In order to apply Proposition 2.2 to the equation (10.41), it remains to estimate

$$R_{N,\theta_{p+1}}(f'(S_{p+1}Y_{p+1}; X_{p+1}))$$

given by (2.38).

• **The estimate of $\|f'(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^1(\delta,N)}$**

It follows from (2.43) that

$$\begin{aligned} \|f'_1(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^1(\delta,N)} & \lesssim \|S_{p+1}\partial_3 Y_{p+1}\|_{L_t^2(H^1)} (\|\partial_3 X_{p+1}\|_{L_t^2(H^{N+6})} \\ & + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\bar{\varepsilon},0} \|\nabla X_{p+1}\|_{0,N+6}) + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\bar{\varepsilon},1} \|\nabla X_{p+1}\|_{0,1} \times \\ & \times (\|S_{p+1}\partial_3 Y_{p+1}\|_{L_t^2(H^{N+6})} + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6} \|S_{p+1}\partial_3 Y_{p+1}\|_{L_t^2(H^3)}) \\ & + (\|S_{p+1}\partial_3 Y_{p+1}\|_{L_t^2(H^{N+6})} + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6} |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\bar{\varepsilon},1}) \|\partial_3 X_{p+1}\|_{L_t^2(H^1)}, \end{aligned}$$

from which, and (P1, $p+1$), (II) of Proposition 2.7 and the fact that $\beta \geq 6\bar{\varepsilon}$, we infer

$$\|f'_1(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^1(\delta,0)} \lesssim \eta \theta_{p+1}^{-\beta+5\bar{\varepsilon}}.$$

While for $-\beta + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}$, it follows from (I) (II) of proposition 2.7 and (P1, $p+1$) that

$$\|f'_1(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^1(\delta,N)} \lesssim \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}(N+5)}.$$

$f'_2(S_{p+1}Y_{p+1}; X_{p+1})$ can be handled along the same line.

For $f'_0(S_{p+1}Y_{p+1}; X_{p+1})$, we deduce from (2.42) that

$$\begin{aligned} \|\langle t \rangle^{\frac{1}{2}} f'_0(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^2(\delta,N)} & \leq \|S_{p+1}\nabla Y_{p+1}\|_{0,0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_{p+1}\|_{L_t^2(H^{N+6})} \\ & + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_{p+1}\|_{L_t^2(L^2)} + \|\langle t \rangle^{\frac{1}{2}} S_{p+1}\nabla \partial_t Y_{p+1}\|_{L_t^2(L^2)} \|\nabla X_{p+1}\|_{0,N+6} \\ & + (\|\langle t \rangle^{\frac{1}{2}} S_{p+1}\nabla \partial_t Y_{p+1}\|_{L_t^2(H^{N+6})} + \|S_{p+1}\nabla Y_{p+1}\|_{0,N+6} |S_{p+1}\nabla \partial_t Y_{p+1}|_{1+\bar{\varepsilon},0}) \|\nabla X_{p+1}\|_{0,0}. \end{aligned}$$

Notice that $f'_0(S_{p+1}Y_{p+1}; X_{p+1})$ is supported in $\{0 \leq t \leq \theta_{p+1}\}$ so that

$$\|f'_0(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^1(\delta,N)} \lesssim (\log \theta_{p+1})^{\frac{1}{2}} \|\langle t \rangle^{\frac{1}{2}} f'_0(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^2(\delta,N)},$$

which together with (P1, $p+1$) and (II) of Proposition 2.7 ensures that

$$(10.44) \quad \|f'(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^1(\delta,0)} \lesssim \eta \theta_{p+1}^{-\beta+6\bar{\varepsilon}} \quad \text{and}$$

$$(10.45) \quad \|f'(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^1(\delta,N)} \lesssim \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}(N+6)} \quad \text{if } -\beta + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}.$$

• **The estimate of $\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} f'(S_{p+1}Y_{p+1}; X_{p+1})\|_{L_t^2(H^{N+1})}$**

It follows from (2.46) that

$$\begin{aligned} & \| \langle t \rangle^{\frac{1}{2}} |D|^{-1} f'_1(S_{p+1}Y_{p+1}; X_{p+1}) \|_{L_t^2(H^{N+1})} \leq |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2},0} (\| \partial_3 X_{p+1} \|_{L_t^2(H^{N+1})} \\ & + |S_{p+1}\nabla Y_{p+1}|_{0,N+1} \| \partial_3 X_{p+1} \|_{L_t^2(H^1)} + (\| \nabla X_{p+1} \|_{0,N+1} + |S_{p+1}\nabla Y_{p+1}|_{0,N+1} \| \nabla X_{p+1} \|_{0,1}) \\ & \times (|S_{p+1}\partial_3 Y_{p+1}|_{\frac{4}{2}+\bar{\varepsilon},1}^{\frac{4}{3}} \| S_{p+1}\partial_3 Y_{p+1} \|_{L_t^2(L^2)}^{\frac{2}{3}} + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\bar{\varepsilon},1}^2) + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2},N+1} \times \\ & \times \left(\| \partial_3 X_{p+1} \|_{L_t^2(H^1)} + (|S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\bar{\varepsilon},0}^{\frac{1}{3}} \| S_{p+1}\partial_3 Y_{p+1} \|_{L_t^2(L^2)}^{\frac{2}{3}} + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{2}+\bar{\varepsilon},0}) \| \nabla X_{p+1} \|_{0,1} \right), \end{aligned}$$

which together with (II) of Proposition 2.7 and (P1, $p+1$) ensures that

$$\| \langle t \rangle^{\frac{1}{2}} |D|^{-1} f'_1(S_{p+1}Y_{p+1}; X_{p+1}) \|_{L_t^2(H^3)} \lesssim \eta \theta_{p+1}^{-\beta+2\bar{\varepsilon}}.$$

For N satisfying $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, we deduce from (I) of Proposition 2.7 and (P1, $p+1$) that

$$\| \langle t \rangle^{\frac{1}{2}} |D|^{-1} f'_1(S_{p+1}Y_{p+1}; X_{p+1}) \|_{L_t^2(H^{N+1})} \lesssim \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}N}.$$

$f'_2(S_{p+1}Y_{p+1}; X_{p+1})$ can be treated similarly.

For $f'_0(S_{p+1}Y_{p+1}; X_{p+1})$, by virtue of (2.45), we get

$$\begin{aligned} & \| \langle t \rangle^{\frac{1}{2}} |D|^{-1} f'_0(S_{p+1}Y_{p+1}; X_{p+1}) \|_{L_t^2(H^{N+1})} \lesssim |S_{p+1}\nabla Y_{p+1}|_{0,0} \| \langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_{p+1} \|_{L_t^2(H^{N+1})} \\ & + |S_{p+1}\nabla Y_{p+1}|_{0,N+1} \| \langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_{p+1} \|_{L_t^2(L^2)} + |S_{p+1}\partial_t Y_{p+1}|_{1+\bar{\varepsilon},1} \| \nabla X_{p+1} \|_{0,N+1} \\ & + (|S_{p+1}\partial_t Y_{p+1}|_{1+\bar{\varepsilon},N+2} + |S_{p+1}\partial_t Y_{p+1}|_{1+\bar{\varepsilon},1} |S_{p+1}\nabla Y_{p+1}|_{0,N+1}) \| \nabla X_{p+1} \|_{0,0}. \end{aligned}$$

As a result, it comes out

$$(10.46) \quad \| \langle t \rangle^{\frac{1}{2}} |D|^{-1} f'(S_{p+1}Y_{p+1}; X_{p+1}) \|_{L_t^2(H^3)} \lesssim \eta \theta_{p+1}^{-\beta+2\bar{\varepsilon}} \quad \text{and}$$

$$(10.47) \quad \| \langle t \rangle^{\frac{1}{2}} |D|^{-1} f'(S_{p+1}Y_{p+1}; X_{p+1}) \|_{L_t^2(H^{N+1})} \lesssim \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}N} \quad \text{if } -\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}.$$

• **The Estimate of** $\| |D|^{-1} f'(S_{p+1}Y_{p+1}; X_{p+1}) \|_{\frac{3}{2}-\delta, N+1}$

By virtue of (2.46), we have

$$\begin{aligned} & \| |D|^{-1} f'_1(S_{p+1}Y_{p+1}; X_{p+1}) \|_{\frac{3}{2}, N+1} \leq |S_{p+1}\partial_3 Y_{p+1}|_{1,0} (|S_{p+1}\nabla Y_{p+1}|_{0,N+1} \| \partial_3 X_{p+1} \|_{\frac{1}{2},1} \\ & + \| \partial_3 X_{p+1} \|_{\frac{1}{2}, N+1}) + (|S_{p+1}\partial_3 Y_{p+1}|_{\frac{4}{7},1}^{\frac{4}{3}} \| S_{p+1}\partial_3 Y_{p+1} \|_{\frac{1}{2},0}^{\frac{2}{3}} + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},1}^2) \times \\ & \times (\| \nabla X_{p+1} \|_{0,N+1} + |S_{p+1}\nabla Y_{p+1}|_{0,N+1} \| \nabla X_{p+1} \|_{0,1}) + |S_{p+1}\partial_3 Y_{p+1}|_{1,N+1} \| \partial_3 X_{p+1} \|_{\frac{1}{2},1} \\ & + (|S_{p+1}\partial_3 Y_{p+1}|_{\frac{7}{8},N+1} |S_{p+1}\partial_3 Y_{p+1}|_{\frac{1}{8},0}^{\frac{1}{3}} \| S_{p+1}\partial_3 Y_{p+1} \|_{\frac{1}{2},0}^{\frac{2}{3}} \\ & + |S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},N+1} |S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},0}) \| \nabla X_{p+1} \|_{0,1}. \end{aligned}$$

Noticing from (2.77) that $\frac{1}{4} - \gamma \geq \bar{\varepsilon}$, so that we get, by applying (II) (i) of Proposition 2.7, that

$$|S_{p+1}\partial_3 Y_{p+1}|_{\frac{7}{8},0} \leq \eta \theta_{p+1}^{\frac{3}{8}-\gamma}, \quad |S_{p+1}\partial_3 Y_{p+1}|_{1,0} \lesssim \eta \theta_{p+1}^{\frac{1}{2}-\gamma}, \quad |S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},0} \leq \eta \theta_{p+1}^{\frac{1}{4}-\gamma}.$$

As a result, it comes out

$$\begin{aligned} (10.48) \quad & \| |D|^{-1} f'_1(S_{p+1}Y_{p+1}; X_{p+1}) \|_{\frac{3}{2},3} \lesssim \eta^2 (\theta_{p+1}^{\frac{1}{2}-\gamma-\beta+3\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-\frac{4}{3}\gamma-\beta+\frac{10}{3}\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-2\gamma-\beta+3\bar{\varepsilon}}) \\ & \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+3\bar{\varepsilon}}, \end{aligned}$$

provided that $\frac{1}{3}\gamma + \frac{2}{3}\beta \geq \frac{1}{3}\bar{\varepsilon}$, which is the case due to (2.77) and (2.76).

For N with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, we deduce from (I) (i) of Corollary 2.7 that

$$|S_{p+1}\partial_3 Y_{p+1}|_{\frac{7}{8},N+1} \leq \eta \theta_{p+1}^{\frac{3}{8}-\gamma+\bar{\varepsilon}(N+1)}, \quad |S_{p+1}\partial_3 Y_{p+1}|_{1,N+1} \lesssim \eta \theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}(N+1)},$$

$$|S_{p+1}\partial_3 Y_{p+1}|_{\frac{3}{4},N+1} \leq \eta \theta_{p+1}^{\frac{1}{4}-\gamma+\bar{\varepsilon}(N+1)}, \quad |S_{p+1}\nabla Y_{p+1}|_{0,N+1} \lesssim \eta \theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+1)},$$

which together with (P1, $p+1$) ensures that

$$(10.49) \quad |||D|^{-1}f'_1(S_{p+1}Y_{p+1}; X_{p+1})|||_{\frac{3}{2},N+1} \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+1)}.$$

Similar estimates as above holds for f'_2 .

To deal with the term $f'_0(S_{p+1}Y_{p+1}; X_{p+1})$, we get, by applying (2.45), that

$$\begin{aligned} |||D|^{-1}f'_0(S_{p+1}Y_{p+1}; X_{p+1})|||_{\frac{3}{2}-\delta,N+1} &\lesssim |S_{p+1}\nabla Y_{p+1}|_{\frac{1}{2}-\delta,0} \|\nabla \partial_t X_{p+1}\|_{1,N+1} \\ &\quad + |S_{p+1}\nabla Y_{p+1}|_{\frac{1}{2}-\delta,N+1} \|\nabla \partial_t X_{p+1}\|_{1,0} + |S_{p+1}\partial_t Y_{p+1}|_{\frac{3}{2}-\delta,1} \|\nabla X_{p+1}\|_{0,N+1} \\ &\quad + (|S_{p+1}\partial_t Y_{p+1}|_{\frac{3}{2}-\delta,N+2} + |S_{p+1}\partial_t Y_{p+1}|_{\frac{3}{2}-\delta,1} |S_{p+1}\nabla Y_{p+1}|_{0,N+1}) \|\nabla X_{p+1}\|_{0,0}. \end{aligned}$$

Then along the same line to proof of (10.48) and (10.49), we can show that

$$(10.50) \quad |||D|^{-1}f'(S_{p+1}Y_{p+1}; X_{p+1})|||_{\frac{3}{2}-\delta,3} \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+4\bar{\varepsilon}},$$

and for N with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, there holds

$$(10.51) \quad |||D|^{-1}f'(S_{p+1}Y_{p+1}; X_{p+1})|||_{\frac{3}{2}-\delta,N+1} \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+2)}.$$

Moreover, we can prove in the same way that

$$(10.52) \quad \begin{aligned} |||D|^{-1}f'(S_p Y_p; X_p)|||_{1,1} &\lesssim \eta^2 \theta_p^{-\beta+2\bar{\varepsilon}}, \\ |||D|^{-1}f'(S_p X_p; X_p)|||_{1,N+1} &\lesssim \eta^2 \theta_p^{-\beta+\bar{\varepsilon}(N+2)} \quad \text{for } -\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}. \end{aligned}$$

Recall (2.38), we get, by summarizing the estimates (10.44), (10.46) and (10.50) that

$$R_{0,\theta_{p+1}}(f'(S_{p+1}Y_{p+1}; X_{p+1})) \lesssim \eta^2 (\theta_{p+1}^{-\beta+6\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-\beta+2\bar{\varepsilon}} + (\log \theta_{p+1}) \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+4\bar{\varepsilon}}) \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma},$$

provided that

$$(10.53) \quad \beta + \frac{1}{2} \geq \gamma + 6\bar{\varepsilon}, \quad \beta \geq \gamma + 2\bar{\varepsilon}, \quad \beta \geq 5\bar{\varepsilon}$$

which is the case here due to (2.77) and (2.76).

While due to (10.53), $-\beta + \bar{\varepsilon}(N_0 + 5) \geq \bar{\varepsilon}$ and $-\gamma + \bar{\varepsilon}(N_0 + 2) \geq \bar{\varepsilon}$, by summarizing the estimates (10.45), (10.47) and (10.51), we achieve

$$R_{N_0,\theta_{p+1}}(f'(S_{p+1}Y_{p+1}; X_{p+1})) \lesssim \eta^2 \theta_{p+1}^{\bar{\varepsilon}(N_0+2)} (\theta_{p+1}^{-\beta+4\bar{\varepsilon}} + \theta_{p+1}^{\frac{1}{2}-\beta} + (\log \theta_{p+1}) \theta_{p+1}^{\frac{1}{2}-\gamma-\beta+2\bar{\varepsilon}}) \lesssim \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N_0}.$$

Now we apply Proposition 2.2 and (VI) of Proposition 2.8 to (10.41), to get

$$|\partial_3 X_{p+1}|_{1,0} + |X_{p+1,t}|_{\frac{3}{2}-\delta,0} + |X_{p+1}|_{\frac{1}{2},0} \leq R_{0,\theta_{p+1}}(f'(S_{p+1}Y_{p+1}; X_{p+1})) + R_{0,\theta_{p+1}}(g_{p+1}) \leq C \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma},$$

and

$$\begin{aligned} |\partial_3 X_{p+1}|_{1,N_0} + |X_{p+1,t}|_{\frac{3}{2}-\delta,N_0} + |X_{p+1}|_{\frac{1}{2},N_0} \\ \leq R_{N_0,\theta_{p+1}}(f'(S_{p+1}Y_{p+1}; X_{p+1})) + R_{N_0,\theta_{p+1}}(g_{p+1}) \leq C \eta^2 \theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N_0}. \end{aligned}$$

Interpolating the above two inequalities gives for all $0 \leq N \leq N_0$,

$$(10.54) \quad |\partial_3 X_{p+1}|_{1,N} + |X_{p+1,t}|_{\frac{3}{2}-\delta,N} + |X_{p+1}|_{\frac{1}{2},N} \lesssim \eta \theta_{p+1}^{\frac{1}{2}-\gamma+\bar{\varepsilon}N}.$$

Whereas it follows from Sobolev embedding Theorem and (P1, $p+1$) that for any $0 \leq N \leq N_0$,

$$(10.55) \quad \begin{aligned} |X_{p+1}|_{0,N} &\lesssim \|\nabla X_{p+1}\|_{0,N+1} \leq \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}N} \leq \eta \theta_{p+1}^{-\gamma+\bar{\varepsilon}N}, \\ |\partial_3 X_{p+1}|_{\frac{1}{2},N} &\lesssim \|\partial_3 X_{p+1}\|_{\frac{1}{2},N+2} \leq \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}(N+1)} \leq \eta \theta_{p+1}^{-\gamma+\bar{\varepsilon}N}, \\ |\partial_t X_{p+1}|_{1-\delta,N} &\lesssim \|\nabla \partial_t X_{p+1}\|_{1,N+1} \leq \eta \theta_{p+1}^{-\beta+\bar{\varepsilon}(N+2)} \leq \eta \theta_{p+1}^{-\gamma+\bar{\varepsilon}N}, \end{aligned}$$

provided that $\beta \geq \gamma + 2\bar{\varepsilon}$, which is satisfied due to (2.77).

By interpolating the inequalities (10.54) and (10.55), we arrive at (P2, $p+1$). This completes the proof of Proposition 2.6 for $p+1$. \square

10.4. The proof of Theorem 2.1. The goal of this subsection is to prove the convergence of the approximate solutions $\{Y_p\}$ constructed via (2.69) in some appropriate norms, which in particular ensures Theorem 2.1.

Proof of Theorem 2.1. We infer from (2.70), (10.52), (P1) of Proposition 2.6 and (V) of Proposition 2.8, that

$$(10.56) \quad \begin{aligned} \|\partial_{tt} X_p\|_{\frac{1}{2},0} &\leq \|(\Delta \partial_t X_p, \partial_3^2 X_p)\|_{\frac{1}{2},0} + \|f'(S_p Y_p; X_p)\|_{\frac{1}{2},0} + \|g_p\|_{\frac{1}{2},0} \leq C \eta \theta_p^{-\beta+2\bar{\varepsilon}}, \\ \|\partial_{tt} X_p\|_{\frac{1}{2},N} &\leq C \eta \theta_p^{-\beta+\bar{\varepsilon}(N+2)}, \quad \text{for } -\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}. \end{aligned}$$

Interpolating the above two inequalities leads to

$$(10.57) \quad \|\partial_{tt} X_p\|_{\frac{1}{2},N} \leq C \eta \theta_p^{-\beta+\bar{\varepsilon}(N+2)}, \quad \forall N \geq 0.$$

Due to the choices of the parameters in (2.77) and (2.76), it follows from (P2) of Proposition 2.6 that

$$\begin{aligned} \sum_{p=0}^{\infty} |\partial_3 Y_{p+1} - \partial_3 Y_p|_{\frac{3}{4}-4\bar{\varepsilon},2} &= \sum_{p=0}^{\infty} |\partial_3 X_p|_{\frac{3}{4}-4\bar{\varepsilon},2} \leq \eta \sum_{p=0}^{\infty} \theta_p^{-\bar{\varepsilon}} < +\infty, \\ \sum_{p=0}^{\infty} |\partial_t Y_{p+1} - \partial_t Y_p|_{\frac{5}{4}-\delta-4\bar{\varepsilon},2} &= \sum_{p=0}^{\infty} |\partial_t X_p|_{\frac{5}{4}-\delta-4\bar{\varepsilon},2} \leq \eta \sum_{p=0}^{\infty} \theta_p^{-\bar{\varepsilon}} < +\infty, \\ \sum_{p=0}^{\infty} |Y_{p+1} - Y_p|_{\frac{1}{4}-4\bar{\varepsilon},2} &= \sum_{p=0}^{\infty} |X_p|_{\frac{1}{4}-4\bar{\varepsilon},2} \leq \eta \sum_{p=0}^{\infty} \theta_p^{-\bar{\varepsilon}} < +\infty. \end{aligned}$$

Similarly, let us take $N_0 = [1/2\bar{\varepsilon}] + 1$ and $N_1 \stackrel{\text{def}}{=} [N_0/2]$, we deduce from (P2) of Proposition 2.6 and (10.56) that

$$\begin{aligned} &\sum_{p=0}^{\infty} \left(\| |D|^{-1} (\partial_3 Y_{p+1} - \partial_3 Y_p, \partial_t Y_{p+1} - \partial_t Y_p) \|_{0,N_1+2} + \|\nabla Y_{p+1} - \nabla Y_p\|_{0,N_1+1} \right. \\ &\quad + \| (\partial_3 Y_{p+1} - \partial_3 Y_p, \langle t \rangle^{\frac{1}{2}} (\nabla \partial_t Y_{p+1} - \nabla \partial_t Y_p)) \|_{L_t^2(H^{N_1+1})} + \|\partial_t Y_{p+1} - \partial_t Y_p\|_{L_t^2(H^{N_1+2})} \\ &\quad + \| (\partial_t Y_{p+1} - \partial_t Y_p, \partial_3 Y_{p+1} - \partial_3 Y_p) \|_{\frac{1}{2},N_1+1} + \|\nabla \partial_t Y_{p+1} - \nabla \partial_t Y_p\|_{1,N_1-1} \\ &\quad \left. + \|\partial_{tt} Y_{p+1} - \partial_{tt} Y_p\|_{\frac{1}{2},N_1-2} \right) \leq \eta \sum_{p=0}^{\infty} \theta_p^{-\beta+\bar{\varepsilon}N_1} \leq \eta \sum_{p=0}^{\infty} \theta_p^{-\bar{\varepsilon}} < +\infty. \end{aligned}$$

This ensures the existence of $Y \in C^2([0, +\infty); C^{N_1-4}(\mathbb{R}^3))$ such that

$$(10.58) \quad |\partial_3 Y - \partial_3 Y_p|_{\frac{3}{4}-4\bar{\varepsilon},2} + |Y_t - \partial_t Y_p|_{\frac{5}{4}-\delta-4\bar{\varepsilon},2} + |Y - Y_p|_{\frac{1}{4}-4\bar{\varepsilon},2} \rightarrow 0,$$

and

$$\begin{aligned}
(10.59) \quad & \| |D|^{-1}(\partial_3 Y - \partial_3 Y_p, Y_t - \partial_t Y_p) \|_{0, N_1+2} + \|\nabla Y - \nabla Y_p\|_{0, N_1+1} + \|\partial_t Y - \partial_t Y_p\|_{L_t^2(H^{N_1+2})} \\
& + \|(\partial_3 Y - \partial_3 Y_p, \langle t \rangle^{\frac{1}{2}}(\nabla \partial_t Y - \nabla \partial_t Y_p))\|_{L_t^2(H^{N_1+1})} + \|\nabla \partial_t Y - \nabla \partial_t Y_p\|_{1, N_1-1} \\
& + \|(\partial_t Y - \partial_t Y_p, \partial_3 Y - \partial_3 Y_p)\|_{\frac{1}{2}, N_1+1} + \|\partial_{tt} Y - \partial_{tt} Y_p\|_{\frac{1}{2}, N_1-2} \rightarrow 0, \quad \text{as } p \rightarrow +\infty,
\end{aligned}$$

which ensures (2.21) and (2.22).

Next we show that Y is the solution to (2.65). As a matter of fact, we first observe from (2.72) and (2.73) that

$$\Phi(Y_{p+1}) - \Phi(Y_0) = \sum_{j=0}^p e_j + \sum_{j=0}^p g_j = E_p + e_p - S_p E_p - S_p \Phi(Y_0),$$

which implies

$$\Phi(Y_{p+1}) = e_p + (1 - S_p)E_p + (1 - S_p)\Phi(Y_0),$$

from which, (10.34), (10.38) and (IV) of Proposition 2.8, we infer

$$(10.60) \quad \|\Phi(Y_{p+1})\|_{1,0} \leq \|e_p\|_{1,0} + \|(1 - S_p)E_p\|_{1,0} + \|(1 - S_p)f(Y_0)\|_{1,0} \leq C\theta_{p+1}^{\delta-\gamma-\beta+2\varepsilon}.$$

Next, we show that $\Phi(Y_{p+1}) \rightarrow \Phi(Y)$ as $p \rightarrow +\infty$ in the norm $\|\cdot\|_{1,0}$. Indeed let us denote $\tilde{\square} \stackrel{\text{def}}{=} \partial_t^2 - \Delta \partial_t - \partial_3^2$, then one has

$$(10.61) \quad \|\Phi(Y) - \Phi(Y_{p+1})\|_{1,0} \leq \|\tilde{\square}(Y - Y_{p+1})\|_{1,0} + \|f(Y) - f(Y_{p+1})\|_{1,0}.$$

Using a Taylor formula, applying (2.45), (2.46), (2.47) and using (10.58), (10.59), we have

$$\begin{aligned}
\|f(Y) - f(Y_{p+1})\|_{1,0} & \leq \int_0^1 \|f'((1-s)Y_{p+1} + sY; Y - Y_{p+1})\|_{1,0} ds \\
& \lesssim C \left(\|\partial_3 Y - \partial_3 Y_{p+1}\|_{\frac{1}{2},1} + \|Y_t - \partial_t Y_{p+1}\|_{\frac{1}{2},1} \right. \\
& \quad \left. + \|\nabla Y_t - \nabla \partial_t Y_{p+1}\|_{1,1} + \|\nabla Y - \nabla Y_{p+1}\|_{0,1} \right) \rightarrow 0, \quad \text{as } p \rightarrow +\infty.
\end{aligned}$$

On the other hand, recall from (2.70) that

$$\tilde{\square} X_p = f'(S_p Y_p; X_p) + g_p,$$

then we get, by applying (P1) of Proposition 2.6, (II) of Proposition 2.7 and (V)(ii) of Proposition 2.8, that

$$\begin{aligned}
\|\tilde{\square} X_p\|_{1,0} & \leq \|f'(S_p Y_p; X_p)\|_{1,0} + \|g_p\|_{1,0} \\
& \lesssim C (\|\partial_3 X_p\|_{\frac{1}{2},1} + \|\partial_t X_p\|_{\frac{1}{2},1} + \|\nabla \partial_t X_p\|_{1,1} + \|\nabla X_p\|_{0,1}) + \|g_p\|_{1,0} \\
& \lesssim C \theta_p^{-\beta+2\varepsilon} + \theta_p^{\delta-\gamma-\beta+2\varepsilon}.
\end{aligned}$$

Consequently, we achieve

$$(10.62) \quad \|\tilde{\square}(Y - Y_{p+1})\|_{1,0} \leq \sum_{j=p+1}^{\infty} \|\tilde{\square} X_j\|_{1,0} \leq C \sum_{j=p+1}^{\infty} \theta_j^{-\beta+2\varepsilon} \rightarrow 0, \quad \text{as } p \rightarrow \infty.$$

We then deduce from (10.61) and (10.62) that

$$\|\Phi(Y) - \Phi(Y_{p+1})\|_{1,0} \rightarrow 0 \quad \text{as } p \rightarrow \infty,$$

which together with (10.60) implies $\Phi(Y) = 0$. Finally, for each p , we have

$$Y_p(0, y) = Y^{(0)}, \quad \partial_t Y_p(0, y) = Y^{(1)}(y),$$

therefore,

$$Y(0, y) = Y^{(0)}, \quad Y_t(0, y) = Y^{(1)}(y),$$

and thus Y is the desired classical solution to (2.65). This ends the proof of Theorem 2.1. \square

APPENDIX A. THE PROOF OF LEMMAS 10.1, 10.2 AND 10.3

The goal of this appendix is to present the proof of Lemmas 10.1, 10.2 and 10.3. Notice that the estimates for $e'_{p,2}$, $e''_{p,2}$ are the same as (even better than) those for $e'_{p,1}$, $e''_{p,1}$, so that we only preform the estimates for the latter in what follows.

A.1. The proof of Lemma 10.1. We divide the proof of this lemma by the following steps:

• The proof of (10.14)

In view of (2.75), we get, by applying (9.2) (with $Y \simeq Y_p + Y_{p+1}$, $X = W = X_p$), that for $N \geq 0$,

$$\begin{aligned} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e''_{p,1} \|_{L_t^2(H^{N+1})} &\lesssim |\partial_3 X_p|_{\frac{1}{2}, N+1} \|\partial_3 X_p\|_{L_t^2(L^2)} + |\partial_3 X_p|_{\frac{1}{2}, 0} \|\partial_3 X_p\|_{L_t^2(H^{N+1})} \\ &+ \sum_{j=p}^{p+1} \left\{ \left(|\nabla Y_j|_{0, N+1} |\partial_3 X_p|_{\frac{1}{2}, 0} + (|\partial_3 Y_j|_{\frac{1}{2}, N+1} + |\nabla Y_j|_{0, N+1} |\partial_3 Y_j|_{\frac{1}{2}, 1}) |\nabla X_p|_{0, 1} \right) \|\partial_3 X_p\|_{L_t^2(H^1)} \right. \\ &+ (|\partial_3 Y_j|_{\frac{1}{2} + \bar{\varepsilon}, 1} + |\partial_3 Y_j|_{\frac{1}{2} + \bar{\varepsilon}, 1}^{\frac{1}{3}} \|\partial_3 Y_j\|_{\frac{1}{2}, 1}^{\frac{2}{3}}) \left(|\nabla X_p|_{0, N+1} \|\partial_3 X_p\|_{L_t^2(L^2)} + |\nabla X_p|_{0, 0} \|\partial_3 X_p\|_{L_t^2(H^{N+1})} \right. \\ &+ |\partial_3 X_p|_{\frac{1}{2}, 1} \|\nabla X_p\|_{0, N+1} + (|\partial_3 X_p|_{\frac{1}{2}, N+1} + |\nabla Y_j|_{0, N+1} |\partial_3 X_p|_{\frac{1}{2}, 1} + |\partial_3 Y_j|_{\frac{1}{2} + \bar{\varepsilon}, 0} |\nabla X_p|_{0, 1}) \|\nabla X_p\|_{0, 1} \Big) \\ &+ (|\partial_3 Y_j|_{\frac{1}{2} + \bar{\varepsilon}, 1}^{\frac{4}{3}} \|\partial_3 Y_j\|_{\frac{1}{2}, 0}^{\frac{2}{3}} + |\partial_3 Y_j|_{\frac{1}{2} + \bar{\varepsilon}, 1}^2) \left(|\nabla X_p|_{0, 1} (\|\nabla X_p\|_{0, N+1} + |\nabla Y_j|_{0, N+1} \|\nabla X_p\|_{0, 1}) \right. \\ &\left. \left. + |\nabla X_p|_{0, N+1} \|\nabla X_p\|_{0, 1} \right) + (|\partial_3 Y_j|_{\frac{1}{2} + \bar{\varepsilon}, N+1} + |\partial_3 Y_j|_{\frac{1}{2} + \bar{\varepsilon}, N+1}^{\frac{1}{3}} \|\partial_3 Y_j\|_{\frac{1}{2}, N+1}^{\frac{2}{3}}) |\partial_3 X_p|_{\frac{1}{2}, 1} \|\nabla X_p\|_{0, 1} \right\}. \end{aligned}$$

Similar estimate holds for $\|\langle t \rangle |D|^{-1} e''_{p,1} \|_{L_t^2(H^{N+1})}$ with $|\partial_3 X_p|_{\frac{1}{2}, l}$ above being replaced by $|\partial_3 X_p|_{1, l}$ and $|\nabla X_p|_{0, l}$ by $|\nabla X_p|_{\frac{1}{2}, l}$.

It follows from (10.7), (10.9) and (10.10) that

$$(A.1) \quad |\partial_3 Y_{p+1}|_{\frac{1}{2} + \bar{\varepsilon}, 1} \leq C\eta, \quad |\nabla Y_{p+1}|_{0, 1} \leq C\eta \quad \text{since } \gamma \geq 3\bar{\varepsilon};$$

$$(A.2) \quad \|\partial_3 Y_{p+1}\|_{\frac{1}{2}, 1} \leq C\eta \quad \text{since } \beta \geq \bar{\varepsilon}.$$

As a result, applying (P1, p) and (P2, p), it comes out

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e''_{p,1} \|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{-\gamma - \beta + \bar{\varepsilon}}, \quad \text{and} \quad \|\langle t \rangle |D|^{-1} e''_{p,1} \|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{\frac{1}{2} - \gamma - \beta + \bar{\varepsilon}}.$$

Interpolating between the above two inequalities gives rise to

$$(A.3) \quad \|\langle t \rangle^{\frac{1}{2} + k} |D|^{-1} e''_{p,1} \|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{k - \gamma - \beta + \bar{\varepsilon}} \quad \text{if } 0 \leq k \leq \frac{1}{2}.$$

While for $0 \leq N \leq N_0 - 1$ such that $-\gamma + \bar{\varepsilon}(N + 1) \geq \bar{\varepsilon}$ and $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$, we deduce from (10.7), (10.9) and (10.10) that

$$(A.4) \quad \begin{aligned} |\partial_3 Y_{p+1}|_{\frac{1}{2} + \bar{\varepsilon}, N+1} &\leq C\eta \theta_{p+1}^{-\gamma + \bar{\varepsilon}(N+2)}, \quad |\nabla Y_{p+1}|_{0, N+1} \leq C\eta \theta_{p+1}^{-\gamma + \bar{\varepsilon}(N+1)} \\ \|\partial_3 Y_{p+1}\|_{\frac{1}{2}, N+1} &\leq C\eta \theta_{p+1}^{-\beta + \bar{\varepsilon}N}. \end{aligned}$$

Therefore for such N , there hold

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e''_{p,1} \|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{-\gamma - \beta + \bar{\varepsilon}(N+1)}, \quad \|\langle t \rangle |D|^{-1} e''_{p,1} \|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{\frac{1}{2} - \gamma - \beta + \bar{\varepsilon}(N+1)}.$$

Interpolating the above two inequalities, we obtain for $0 \leq k \leq \frac{1}{2}$, $N \leq N_0 - 1$ such that $-\gamma + \bar{\varepsilon}(N + 1) \geq \bar{\varepsilon}$ and $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$,

$$(A.5) \quad \|\langle t \rangle^{\frac{1}{2} + k} |D|^{-1} e''_{p,1} \|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k - \gamma - \beta + \bar{\varepsilon}(N+1)}.$$

Interpolating between (A.3) and (A.5) leads to (10.14).

• The proof of (10.15)

In order to do so, we get by applying (9.2) (with $Y \simeq Y_p$, $X = (1 - S_p)Y_p$, $W = X_p$) that for $N \geq 0$,

$$\begin{aligned}
\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e'_{p,1} \|_{L_t^2(H^{N+1})} &\lesssim \left(|(1 - S_p) \partial_3 Y_p|_{\frac{1}{2}, N+1} + |\nabla Y_p|_{0, N+1} |(1 - S_p) \partial_3 Y_p|_{\frac{1}{2}, 0} \right. \\
&\quad \left. + (|\partial_3 Y_p|_{\frac{1}{2}, N+1} + |\nabla Y_p|_{0, N+1} |\partial_3 Y_p|_{\frac{1}{2}, 1}) |(1 - S_p) \nabla Y_p|_{0, 1} \right) \|\partial_3 X_p\|_{L_t^2(H^1)} \\
&\quad + |(1 - S_p) \partial_3 Y_p|_{\frac{1}{2}, 0} \|\partial_3 X_p\|_{L_t^2(H^{N+1})} + (|\partial_3 Y_p|_{\frac{1}{2} + \bar{\varepsilon}, 1} + |\partial_3 Y_p|_{\frac{1}{2} + \bar{\varepsilon}, 1}^{\frac{1}{3}} \|\partial_3 Y_p\|_{\frac{1}{2}, 1}^{\frac{2}{3}}) \\
&\quad \times \left(|(1 - S_p) \nabla Y_p|_{0, 1} (\|\partial_3 X_p\|_{L_t^2(H^{N+1})} + |\partial_3 Y_p|_{\frac{1}{2}, N+1} \|\nabla X_p\|_{0, 1}) \right. \\
&\quad \left. + |(1 - S_p) \partial_3 Y_p|_{\frac{1}{2}, N+1} \|\nabla X_p\|_{0, 1} + |(1 - S_p) \partial_3 Y_p|_{\frac{1}{2}, 1} (\|\nabla X_p\|_{0, N+1} + |\nabla Y_p|_{0, N+1} \|\nabla X_p\|_{0, 1}) \right. \\
&\quad \left. + |(1 - S_p) \nabla Y_p|_{0, N+1} \|\partial_3 X_p\|_{L_t^2(L^2)} \right) + (|\partial_3 Y_p|_{\frac{1}{2} + \bar{\varepsilon}, 1}^{\frac{4}{3}} \|\partial_3 Y_p\|_{\frac{1}{2}, 0}^{\frac{2}{3}} + |\partial_3 Y_p|_{\frac{1}{2} + \bar{\varepsilon}, 1}^2) \\
&\quad \times \left(|(1 - S_p) \nabla Y_p|_{0, N+1} \|\nabla X_p\|_{0, 1} + |(1 - S_p) \nabla Y_p|_{0, 1} (\|\nabla X_p\|_{0, N+1} + |\nabla Y_p|_{0, N+1} \|\nabla X_p\|_{0, 1}) \right) \\
&\quad + (|\partial_3 Y_p|_{\frac{1}{2} + \bar{\varepsilon}, N+1} + |\partial_3 Y_p|_{\frac{1}{2} + \bar{\varepsilon}, N+1}^{\frac{1}{3}} \|\partial_3 Y_p\|_{\frac{1}{2}, N+1}^{\frac{2}{3}}) |(1 - S_p) \partial_3 Y_p|_{\frac{1}{2}, 1} \|\nabla X_p\|_{0, 1}
\end{aligned}$$

Similar estimate for $\|\langle t \rangle |D|^{-1} e'_{p,1} \|_{L_t^2(H^{N+1})}$ holds with $|(1 - S_p) \partial_3 Y_p|_{\frac{1}{2}, l}$ above being replaced by $|(1 - S_p) \partial_3 Y_p|_{1, l}$ and $|(1 - S_p) \nabla Y_p|_{\frac{1}{2}, l}$ by $|(1 - S_p) \nabla Y_p|_{0, l}$.

So that by virtue of (A.1), (A.2) and (III) of Proposition 2.7, we infer that

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e'_{p,1} \|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{-\gamma - \beta + \bar{\varepsilon}}, \quad \|\langle t \rangle |D|^{-1} e'_{p,1} \|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{\frac{1}{2} - \gamma - \beta + \bar{\varepsilon}}.$$

Interpolating the above two inequalities, we obtain

$$(A.6) \quad \|\langle t \rangle^{\frac{1}{2} + k} |D|^{-1} e'_{p,1} \|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{k - \gamma - \beta + 2\bar{\varepsilon}} \quad \text{for } 0 \leq k \leq \frac{1}{2}.$$

Note that for $N \leq N_0 - 1$ such that $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$ and $-\gamma + \bar{\varepsilon}(N + 1) \geq \bar{\varepsilon}$, (A.4) holds. And hence we have

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e'_{p,1} \|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{-\gamma - \beta + \bar{\varepsilon}(N+1)}, \quad \|\langle t \rangle |D|^{-1} e'_{p,1} \|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{\frac{1}{2} - \gamma - \beta + \bar{\varepsilon}(N+1)}.$$

Interpolating the above inequalities gives for such N and $0 \leq k \leq \frac{1}{2}$ that

$$(A.7) \quad \|\langle t \rangle^{\frac{1}{2} + k} |D|^{-1} e'_{p,1} \|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k - \gamma - \beta + \bar{\varepsilon}(N+1)}.$$

Interpolating between (A.6) and (A.7) leads to (10.15).

• The proof of (10.16)

It follows from (9.1) and (2.75) that

$$\begin{aligned}
\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e''_{p,0} \|_{L_t^2(H^{N+1})} &\lesssim |\nabla X_p|_{0, N+1} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p \|_{L_t^2(L^2)} \\
&\quad + |\nabla X_p|_{0, 0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p \|_{L_t^2(H^{N+1})} + |\partial_t X_p|_{1 + \bar{\varepsilon}, N+2} \|\nabla X_p\|_{0, 0} + |\partial_t X_p|_{1 + \bar{\varepsilon}, 1} \|\nabla X_p\|_{0, N+1} \\
&\quad + \sum_{j=p}^{p+1} \left(|\partial_t Y_j|_{1 + \bar{\varepsilon}, 1} (|\nabla X_p|_{0, N+1} \|\nabla X_p\|_{0, 0} + |\nabla X_p|_{0, 0} \|\nabla X_p\|_{0, N+1}) \right. \\
&\quad \left. + (|\partial_t Y_j|_{1 + \bar{\varepsilon}, N+2} + |\nabla Y_j|_{0, N+1} |\partial_t Y_j|_{1 + \bar{\varepsilon}, 1}) |\nabla X_p|_{0, 0} \|\nabla X_p\|_{0, 0} \right. \\
&\quad \left. + |\nabla Y_j|_{0, N+1} (|\nabla X_p|_{0, 0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p \|_{L_t^2(L^2)} + |\partial_t X_p|_{1 + \bar{\varepsilon}, 1} \|\nabla X_p\|_{0, 0}) \right).
\end{aligned}$$

Recall that $\alpha = \frac{1}{2} - \delta - \bar{\varepsilon} < \frac{1}{2} - \delta$, similar estimate for $\|\langle t \rangle^{\frac{1}{2}+\alpha} |D|^{-1} e''_{p,0}\|_{L_t^2(H^{N+1})}$ holds with $|\nabla X_p|_{0,l}$ above replaced by $|\nabla X_p|_{\alpha,l}$ and $|\partial_t X_p|_{1+\bar{\varepsilon},l}$ above by $|\partial_t X_p|_{\frac{3}{2}-\delta,l}$.

Note from (2.76) that $\gamma \geq \delta + 4\bar{\varepsilon}$, so that we deduce from (10.8) that,

$$(A.8) \quad |\partial_t Y_{p+1}|_{1+\bar{\varepsilon},2} \leq C\eta,$$

which implies

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e''_{p,0}\|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+3\bar{\varepsilon}}, \quad \|\langle t \rangle^{\alpha+\frac{1}{2}} |D|^{-1} e''_{p,0}\|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+2\bar{\varepsilon}}.$$

Interpolating between the above two inequalities yields

$$(A.9) \quad \|\langle t \rangle^{\frac{1}{2}+k} |D|^{-1} e''_{p,0}\|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+3\bar{\varepsilon}} \quad \text{for } 0 \leq k \leq \alpha.$$

For $N \leq N_0 - 2$ satisfying $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, we deduce from (10.8) and (10.9) that

$$(A.10) \quad |\partial_t Y_{p+1}|_{1+\bar{\varepsilon},N+2} \leq C\eta \theta_{p+1}^{\delta-\gamma+\bar{\varepsilon}(N+3)}, \quad |\nabla Y_{p+1}|_{0,N+1} \leq C\eta \theta_{p+1}^{-\gamma+\bar{\varepsilon}(N+1)},$$

so that for such N , we have

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e''_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}, \quad \|\langle t \rangle^{\alpha+\frac{1}{2}} |D|^{-1} e''_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+2)}.$$

By interpolating between the above inequalities, we achieve for such N and $0 \leq k \leq \alpha$ that

$$(A.11) \quad \|\langle t \rangle^{\frac{1}{2}+k} |D|^{-1} e''_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}.$$

Interpolating between (A.9) and (A.11) leads to (10.16).

• The proof of (10.17)

Again in view of (2.75), we deduce from (9.1) that

$$\begin{aligned} \|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e'_{p,0}\|_{L_t^2(H^{N+1})} &\lesssim |(1-S_p) \nabla Y_p|_{0,N+1} (\|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L_t^2(L^2)} + |\partial_t Y_p|_{1+\bar{\varepsilon},1} \|\nabla X_p\|_{0,0}) \\ &\quad + |(1-S_p) \partial_t Y_p|_{1+\bar{\varepsilon},N+2} \|\nabla X_p\|_{0,0} + |(1-S_p) \partial_t Y_p|_{1+\bar{\varepsilon},1} (|\nabla Y_p|_{0,N+1} \|\nabla X_p\|_{0,0} + \|\nabla X_p\|_{0,N+1}) \\ &\quad + |(1-S_p) \nabla Y_p|_{0,0} \left(|\partial_t Y_p|_{1+\bar{\varepsilon},N+2} + |\nabla Y_p|_{0,N+1} |\partial_t Y_p|_{1+\bar{\varepsilon},1} \right) \|\nabla X_p\|_{0,0} \\ &\quad + |\nabla Y_p|_{0,N+1} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L_t^2(L^2)} + \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L_t^2(H^{N+1})} + \|\nabla X_p\|_{0,N+1} |\partial_t Y_p|_{1+\bar{\varepsilon},1} \Big). \end{aligned}$$

A similar estimate holds for $\|\langle t \rangle^{\frac{1}{2}+\alpha} |D|^{-1} e'_{p,0}\|_{L_t^2(H^{N+1})}$ with $|(1-S_p) \nabla Y_p|_{0,l}$ above being replaced by $|(1-S_p) \nabla Y_p|_{\alpha,l}$ and $|(1-S_p) \partial_t Y_p|_{1+\bar{\varepsilon},l}$ above by $|(1-S_p) \partial_t Y_p|_{\frac{3}{2}-\delta,l}$.

Hence by virtue of (A.8), we have

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e'_{p,0}\|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+3\bar{\varepsilon}}, \quad \|\langle t \rangle^{\frac{1}{2}+\alpha} |D|^{-1} e'_{p,0}\|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+2\bar{\varepsilon}}.$$

Interpolating between the above inequalities gives

$$(A.12) \quad \|\langle t \rangle^{\frac{1}{2}+k} |D|^{-1} e'_{p,0}\|_{L_t^2(H^1)} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+3\bar{\varepsilon}} \quad \text{for } 0 \leq k \leq \alpha.$$

Whereas for $N \leq N_0 - 2$ such that $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, we infer from (A.10) that

$$\|\langle t \rangle^{\frac{1}{2}} |D|^{-1} e'_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}, \quad \|\langle t \rangle^{\frac{1}{2}+\alpha} |D|^{-1} e'_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+2)}.$$

Interpolating between the above inequalities, we achieve for such N and $0 \leq k \leq \alpha$ that

$$(A.13) \quad \|\langle t \rangle^{\frac{1}{2}+k} |D|^{-1} e'_{p,0}\|_{L_t^2(H^{N+1})} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+3)}.$$

Interpolating between (A.12) and (A.13) leads to (10.17). The proof of Lemma 10.1 is complete.

A.2. The proof of Lemma 10.2. As in the previous lemma, we shall divide the proof of this lemma into the following steps:

• The proof of (10.18)

Thanks to (2.75), we get, by applying (9.2) that for $N \geq 0$

$$\begin{aligned} |||D|^{-1}e''_{p,1}|||_{\frac{3}{2},N+1} &\lesssim |\partial_3 X_p|_{1,N+1} + |\partial_3 X_p|_{1,0} \|\partial_3 X_p\|_{\frac{1}{2},N+1} + \sum_{j=p}^{p+1} \left\{ \|\partial_3 X_p\|_{\frac{1}{2},1} \times \right. \\ &\quad \times \left(|\nabla Y_j|_{0,N+1} |\partial_3 X_p|_{1,0} + (|\partial_3 Y_j|_{\frac{1}{2},N+1} \|\partial_3 X_p\|_{\frac{1}{2},0} + |\nabla Y_j|_{0,N+1} |\partial_3 Y_j|_{\frac{1}{2},1}) |\nabla X_p|_{\frac{1}{2},1} \right) \\ &\quad + (|\partial_3 Y_j|_{\frac{1}{2},1} + |\partial_3 Y_j|_{\frac{1}{2},1}^{\frac{1}{3}} \|\partial_3 Y_j\|_{\frac{3}{2},1}^{\frac{2}{3}}) \left(|\nabla X_p|_{\frac{1}{2},N+1} \|\partial_3 X_p\|_{\frac{1}{2},0} + |\nabla X_p|_{\frac{1}{2},0} \|\partial_3 X_p\|_{\frac{1}{2},N+1} \right. \\ &\quad + (|\partial_3 Y_j|_{\frac{1}{2},N+1} |\nabla X_p|_{\frac{1}{2},1} + |\partial_3 X_p|_{1,N+1}) \|\nabla X_p\|_{0,1} + |\partial_3 X_p|_{1,1} (\|\nabla X_p\|_{0,N+1} \\ &\quad + |\nabla Y_j|_{0,N+1} \|\nabla X_p\|_{0,1}) \Big) + (|\partial_3 Y_j|_{\frac{1}{2},1}^{\frac{4}{3}} \|\partial_3 Y_j\|_{\frac{3}{2},0}^{\frac{2}{3}} + |\partial_3 Y_j|_1^2) \times \\ &\quad \times \left(|\nabla X_p|_{\frac{1}{2},1} \|\nabla X_p\|_{0,N+1} + (|\nabla X_p|_{\frac{1}{2},N+1} + |\nabla Y_j|_{0,N+1} |\nabla X_p|_{\frac{1}{2},1}) \|\nabla X_p\|_{0,1} \right) \\ &\quad \left. + (|\partial_3 Y_j|_{\frac{1}{2},N+1} + |\partial_3 Y_j|_{\frac{1}{2},N+1}^{\frac{1}{3}} \|\partial_3 Y_j\|_{\frac{3}{2},N+1}^{\frac{2}{3}}) |\partial_3 X_p|_{1,1} \|\nabla X_p\|_{0,1} \right\}. \end{aligned}$$

A similar estimate holds for $|||D|^{-1}e''_{p,1}|||_{1,N+1}$ with $|\partial_3 X_p|_{1,l}$ and $|\nabla X_p|_{\frac{1}{2},l}$ above being replaced by $|\partial_3 X_p|_{\frac{1}{2},l}$ and $|\nabla X_p|_{0,l}$ respectively.

Hence it follows from (A.1), (P1, p) and (P2, p) that

$$|||D|^{-1}e''_{p,1}|||_{1,1} \lesssim \eta^2 \theta_p^{-\gamma-\beta+\bar{\varepsilon}}, \quad |||D|^{-1}e''_{p,1}|||_{\frac{3}{2},1} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}}.$$

Interpolating the above two inequalities yields

$$(A.14) \quad |||D|^{-1}e''_{p,1}|||_{1+k,1} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}} \quad \text{for } 0 \leq k \leq \frac{1}{2}.$$

Whereas for $N \leq N_0 - 1$ with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$ and $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$, (A.4) holds, we infer that

$$|||D|^{-1}e''_{p,1}|||_{\frac{3}{2},N+1} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+1)}, \quad |||D|^{-1}e''_{p,1}|||_{1,N+1} \lesssim \eta^2 \theta_p^{-\gamma-\beta+\bar{\varepsilon}(N+1)}.$$

Interpolating the above two inequalities leads to

$$(A.15) \quad |||D|^{-1}e''_{p,1}|||_{1+k,N+1} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)}$$

for $0 \leq k \leq \frac{1}{2}$, and $N \leq N_0 - 1$ with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$ and $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$.

Interpolating between (A.14) and (A.15) gives rise to (10.18).

• The proof of (10.19)

Applying (9.2) to $e'_{p,1}$ determined by (2.75) gives that for $N \geq 0$,

$$\begin{aligned} |||D|^{-1}e'_{p,1}|||_{\frac{3}{2},N+1} &\lesssim |(1-S_p)\partial_3 Y_p|_{1,0}(\|\partial_3 X_p\|_{\frac{1}{2},N+1} + |\nabla Y_p|_{0,N+1}\|\partial_3 X_p\|_{\frac{1}{2},0}) \\ &+ |(1-S_p)\partial_3 Y_p|_{1,N+1}\|\partial_3 X_p\|_{\frac{1}{2},0} + (|\partial_3 Y_p|_{\frac{1}{2},1} + |\partial_3 Y_p|_{\frac{1}{2},1}^{\frac{1}{3}}\|\partial_3 Y_p\|_{\frac{2}{3},1}^{\frac{2}{3}}) \times \\ &\times \left(|(1-S_p)\nabla Y_p|_{\frac{1}{2},N+1}\|\partial_3 X_p\|_{\frac{1}{2},0} + |(1-S_p)\nabla Y_p|_{\frac{1}{2},1}(\|\partial_3 X_p\|_{\frac{1}{2},N+1} + |\partial_3 Y_p|_{\frac{1}{2},N+1}\|\nabla X_p\|_{0,1}) \right. \\ &+ |(1-S_p)\partial_3 Y_p|_{1,N+1}\|\nabla X_p\|_{0,1} + |(1-S_p)\partial_3 Y_p|_{1,1}(\|\nabla X_p\|_{0,N+1} + |\nabla Y_p|_{0,N+1}\|\nabla X_p\|_{0,1}) \Big) \\ &+ (|\partial_3 Y_p|_{\frac{1}{2},N+1} + |\nabla Y_p|_{0,N+1}|\partial_3 Y_p|_{\frac{1}{2},1})|(1-S_p)\nabla Y_p|_{\frac{1}{2},1}\|\partial_3 X_p\|_{\frac{1}{2},1} \\ &+ (|\partial_3 Y_p|_{\frac{1}{2},N+1} + |\partial_3 Y_p|_{\frac{1}{2},N+1}^{\frac{1}{3}}\|\partial_3 Y_p\|_{\frac{2}{3},N+1}^{\frac{2}{3}})|(1-S_p)\partial_3 Y_p|_{1,1}\|\nabla X_p\|_{0,1} \\ &+ (|\partial_3 Y_p|_{\frac{1}{2},1}^{\frac{4}{3}}\|\partial_3 Y_p\|_{\frac{2}{3},0}^{\frac{2}{3}} + |\partial_3 Y_p|_{\frac{1}{2},1}^2) \Big(|(1-S_p)\nabla Y_p|_{\frac{1}{2},N+1}\|\nabla X_p\|_{0,1} \\ &\quad + |(1-S_p)\nabla Y_p|_{\frac{1}{2},1}(\|\nabla X_p\|_{0,N+1} + |\nabla Y_p|_{0,N+1}\|\nabla X_p\|_{0,1}) \Big). \end{aligned}$$

A similar estimate holds for $|||D|^{-1}e'_{p,1}|||_{1,N+1}$ with $|(1-S_p)\partial_3 X_p|_{1,l}$ and $|(1-S_p)\nabla X_p|_{\frac{1}{2},l}$ above being replaced by $|(1-S_p)\partial_3 Y_p|_{\frac{1}{2},l}$ and $|(1-S_p)\nabla X_p|_{0,l}$ respectively.

Hence we deduce from (A.1) that

$$|||D|^{-1}e'_{p,1}|||_{1,1} \lesssim \eta^2 \theta_p^{-\gamma-\beta+\bar{\varepsilon}}, \quad |||D|^{-1}e'_{p,1}|||_{\frac{3}{2},1} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}}.$$

Interpolating the above two inequalities yields

$$(A.16) \quad |||D|^{-1}e'_{p,1}|||_{1+k,1} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}} \quad \text{for } 0 \leq k \leq \frac{1}{2}.$$

For $N \leq N_0 - 1$ satisfying $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$ and $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, (A.4) holds, so that we infer that

$$|||D|^{-1}e'_{p,1}|||_{1,N+1} \lesssim \eta^2 \theta_p^{-\gamma-\beta+\bar{\varepsilon}(N+1)}, \quad |||D|^{-1}e'_{p,1}|||_{\frac{3}{2},N+1} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+1)}.$$

Interpolating the above inequalities leads to

$$(A.17) \quad |||D|^{-1}e'_{p,1}|||_{1+k,N+1} \lesssim \eta^2 \theta_p^{k-\gamma-\beta+\bar{\varepsilon}(N+1)},$$

for $0 \leq k \leq \frac{1}{2}$, $N \leq N_0 - 1$ such that $-\beta + \bar{\varepsilon}N \geq \bar{\varepsilon}$ and $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$.

We then conclude the proof of (10.19) by interpolating between (A.16) and (A.17).

• The proof of (10.20)

Applying (9.1) $e''_{p,0}$, which is determined by (2.75) gives that for $N \geq 0$,

$$\begin{aligned} |||D|^{-1}e''_{p,0}|||_{1,N+1} &\lesssim |\nabla X_p|_{0,N+1}\|\nabla \partial_t X_p\|_{1,0} + |\nabla X_p|_{0,0}\|\nabla \partial_t X_p\|_{1,N+1} + |\partial_t X_p|_{1,N+2}\|\nabla X_p\|_{0,0} \\ &+ |\partial_t X_p|_{1,1}\|\nabla X_p\|_{0,N+1} + \sum_{j=p}^{p+1} \Big(|\partial_t Y_j|_{1,1}(|\nabla X_p|_{0,N+1}\|\nabla X_p\|_{0,0} + |\nabla X_p|_{0,0}\|\nabla X_p\|_{0,N+1}) \\ &+ (|\partial_t Y_j|_{1,N+2} + |\nabla Y_j|_{0,N+1}|\partial_t Y_j|_{1,1})|\nabla X_p|_{0,0}\|\nabla X_p\|_{0,0} \\ &\quad + |\nabla Y_j|_{0,N+1}(|\nabla X_p|_{0,0}\|\nabla \partial_t X_p\|_{1,0} + |\partial_t X_p|_{1,1}\|\nabla X_p\|_{0,0}) \Big). \end{aligned}$$

An similar estimate holds for $|||D|^{-1}e''_{p,0}|||_{\frac{3}{2}-\delta,N+1}$ with $|\nabla X_p|_{0,l}$ and $|\partial_t X_p|_{1,l}$ being replaced by $|\nabla X_p|_{\frac{1}{2}-\delta,l}$ and $|\partial_t X_p|_{\frac{3}{2}-\delta,l}$ respectively.

Therefore,

$$|||D|^{-1}e''_{p,0}|||_{1,1} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+2\bar{\varepsilon}}, \quad |||D|^{-1}e''_{p,0}|||_{\frac{3}{2}-\delta,1} \lesssim \eta^2 \theta_p^{-\gamma-\beta+2\bar{\varepsilon}}.$$

By interpolating between the above two inequalities, we obtain

$$(A.18) \quad |||D|^{-1}e''_{p,0}|||_{1+k,1} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+2\bar{\varepsilon}} \quad \text{for } 0 \leq k \leq \frac{1}{2} - \delta.$$

For $N \leq N_0 - 2$ with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, we get, by applying (A.10), that

$$|||D|^{-1}e''_{p,0}|||_{1,N+1} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}, \quad |||D|^{-1}e''_{p,0}|||_{\frac{3}{2}-\delta,N+1} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+2)}.$$

Interpolating the above two inequalities, we obtain

$$(A.19) \quad |||D|^{-1}e''_{p,0}|||_{1+k,N+1} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}$$

for $0 \leq k \leq \frac{1}{2} - \delta$, $N \leq N_0 - 2$ satisfying $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$.

By interpolating the inequalities (A.18) and (A.19), we conclude the proof of (10.20).

• The proof of (10.21)

Again by applying (9.1) $e'_{p,0}$ determined by (2.75), we obtain for $N \geq 0$

$$\begin{aligned} |||D|^{-1}e'_{p,0}|||_{1,N+1} &\lesssim |(1-S_p)\nabla Y_p|_{0,0} \left((|\partial_t Y_p|_{1,N+2} + |\nabla Y_p|_{0,N+1} |\partial_t Y_p|_{1,1}) \|\nabla X_p\|_{0,0} \right. \\ &\quad \left. + |\partial_t Y_p|_{1,1} \|\nabla X_p\|_{0,N+1} + \|\nabla \partial_t X_p\|_{1,N+1} + |\nabla Y_p|_{0,N+1} \|\nabla \partial_t X_p\|_{1,0} \right) \\ &\quad + |(1-S_p)\nabla Y_p|_{0,N+1} (|\partial_t Y_p|_{1,1} \|\nabla X_p\|_{0,0} + \|\nabla \partial_t X_p\|_{1,0}) \\ &\quad + |(1-S_p)\partial_t Y_p|_{1,N+2} \|\nabla X_p\|_{0,0} + |(1-S_p)\partial_t Y_p|_{1,1} (\|\nabla X_p\|_{0,N+1} + |\nabla Y_p|_{0,N+1} \|\nabla X_p\|_{0,0}). \end{aligned}$$

A similar estimate holds for $|||D|^{-1}e'_{p,0}|||_{\frac{3}{2}-\delta,N+1}$ with $|(1-S_p)\nabla Y_p|_{0,l}$ and $|(1-S_p)\partial_t Y_p|_{1,l}$ being replaced by $|(1-S_p)\nabla Y_p|_{\frac{1}{2}-\delta,l}$ and $|(1-S_p)\partial_t Y_p|_{\frac{3}{2}-\delta,l}$ respectively.

So that it follows from (A.8) that

$$|||D|^{-1}e'_{p,0}|||_{1,1} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+2\bar{\varepsilon}}, \quad |||D|^{-1}e'_{p,0}|||_{\frac{3}{2}-\delta,1} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+2\bar{\varepsilon}}.$$

Interpolating the above two inequalities yields

$$(A.20) \quad |||D|^{-1}e'_{p,0}|||_{1+k,1} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+2\bar{\varepsilon}} \quad \text{for } 0 \leq k \leq \frac{1}{2} - \delta.$$

For $N \leq N_0 - 2$ with $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$, we deduce from (A.10) that

$$|||D|^{-1}e'_{p,0}|||_{1,N+1} \lesssim \eta^2 \theta_p^{\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}, \quad |||D|^{-1}e'_{p,0}|||_{\frac{3}{2}-\delta,N+1} \lesssim \eta^2 \theta_p^{\frac{1}{2}-\gamma-\beta+\bar{\varepsilon}(N+2)}.$$

Interpolating the above two inequalities yields

$$(A.21) \quad |||D|^{-1}e'_{p,0}|||_{1+k,N+1} \lesssim \eta^2 \theta_p^{k+\delta-\gamma-\beta+\bar{\varepsilon}(N+2)}$$

for $0 \leq k \leq \frac{1}{2} - \delta$, $N \leq N_0 - 2$ such that $-\gamma + \bar{\varepsilon}(N+1) \geq \bar{\varepsilon}$. By interpolating the inequalities (A.20), (A.21), we obtain (10.21).

This completes the proof of Lemma 10.2.

A.3. The proof of Lemma 10.3. We divide the proof of this lemma by the following steps:

• The proof of (10.22)

Applying (9.15) to $e''_{p,1}$ (with $Y \simeq Y_p + Y_{p+1}$, $X = W = X_p$), which is determined by (2.75), yields that for any $N \geq 0$,

$$\begin{aligned} \|e''_{p,1}\|_{L_t^1(\delta,N)} &\lesssim \|\partial_3 X_p\|_{L_t^2(H^{N+6})} \|\partial_3 X_p\|_{L_t^2(L^2)} + \sum_{j=p}^{p+1} \|\partial_3 X_p\|_{L_t^2(H^1)} \left\{ \left(\|\nabla Y_j\|_{0,N+6} |\partial_3 X_p|_{\frac{1}{2}+\bar{\varepsilon},0} \right. \right. \\ &\quad \left. \left. + |\partial_3 Y_j|_{\frac{1}{2}+\bar{\varepsilon},0} \|\nabla X_p\|_{0,N+6} + \left(\|\partial_3 Y_j\|_{L_t^2(H^{N+6})} + \|\nabla Y_j\|_{0,N+6} |\partial_3 Y_j|_{\frac{1}{2}+\bar{\varepsilon},1} \right) |\nabla X_p|_{0,1} \right) \\ &\quad + \left(|\partial_3 Y_j|_{\frac{1}{2}+\bar{\varepsilon},0} \|\nabla X_p\|_{0,0} + \|\partial_3 Y_j\|_{L_t^2(L^2)} |\nabla X_p|_{0,0} + |\partial_3 Y_j|_{\frac{1}{2}+\bar{\varepsilon},0} \|\nabla X_p\|_{0,0} \right) \|\partial_3 X_p\|_{L_t^2(H^{N+6})} \\ &\quad + \left(\|\partial_3 Y_j\|_{L_t^2(H^{N+6})} + \|\nabla Y_j\|_{0,N+6} (|\partial_3 Y_j|_{\frac{1}{2}+\bar{\varepsilon},0} + \|\partial_3 Y_j\|_{L_t^2(H^3)}) \right) \|\nabla X_p\|_{0,1} |\partial_3 X_p|_{\frac{1}{2}+\bar{\varepsilon},1} \\ &\quad + |\partial_3 Y_j|_{\frac{1}{2}+\bar{\varepsilon},1} \|\partial_3 Y_p\|_{L_t^2(H^3)} \|\nabla X_p\|_{0,N+6} \|\nabla X_p\|_{0,0} \\ &\quad \left. + \left(|\partial_3 Y_j|_{\frac{1}{2}+\bar{\varepsilon},1} \|\partial_3 Y_j\|_{L_t^2(H^{N+6})} + \|\nabla Y_j\|_{0,N+6} |\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},1} \|\partial_3 Y_j\|_{L_t^2(H^3)} \right) |\nabla X_p|_{0,0} \|\nabla X_p\|_{0,1} \right\}. \end{aligned}$$

Note from (2.77) and (2.76) that $\beta \geq 6\bar{\varepsilon}$, so that we deduce from (10.10) that

$$(A.22) \quad \|\partial_3 Y_{p+1}\|_{0,6} + \|\nabla Y_{p+1}\|_{0,6} + \|\partial_3 Y_{p+1}\|_{L_t^2(H^6)} \leq C\eta.$$

As a result, it comes out

$$(A.23) \quad \|e''_{p,1}\|_{L_t^1(\delta,0)} \lesssim \eta^2 (\theta_p^{-2\beta+5\bar{\varepsilon}} + \theta_p^{-\gamma-\beta+5\bar{\varepsilon}}) \lesssim \eta^2 \theta_p^{-\beta-\gamma+5\bar{\varepsilon}}.$$

The case when $N \leq N_0 - 6$ with $-\beta + \bar{\varepsilon}(N+5) \geq \bar{\varepsilon}$, it follows from (10.10) that

$$(A.24) \quad \|\nabla Y_{p+1}\|_{0,N+6} + \|\partial_3 Y_{p+1}\|_{L_t^2(H^{N+6})} \leq C\eta \theta_p^{-\beta+\bar{\varepsilon}(N+5)},$$

hence, we achieve

$$(A.25) \quad \|e''_{p,1}\|_{L_t^1(\delta,N)} \lesssim \eta^2 (\theta_p^{-2\beta+\bar{\varepsilon}(N+5)} + \theta_p^{-\gamma-\beta+\bar{\varepsilon}(N+5)}) \lesssim \eta^2 \theta_p^{-\beta-\gamma+\bar{\varepsilon}(N+5)}.$$

(10.22) then follows by interpolating (A.23) and (A.25).

• The proof of (10.23)

By applying (9.15) to $e'_{p,1}$ (with $Y \simeq Y_p$, $X = (1 - S_p)Y_p$ and $W = X_p$) and noticing that

$$\|(1 - S_p)\partial_3 Y\|_N \leq \|\partial_3 Y\|_N, \quad \|(1 - S_p)\nabla Y_p\|_N \leq \|\nabla Y\|_N$$

we get

$$\begin{aligned} \|e'_{p,1}\|_{L_t^1(\delta,N)} &\lesssim \left(\|\partial_3 Y_p\|_{L_t^2(H^{N+6})} (1 + |(1 - S_p)\nabla Y_p|_{0,1}) + \|\nabla Y_p\|_{0,N+6} |(1 - S_p)\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},0} \right. \\ &\quad \left. + |\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},0} \|\nabla Y_p\|_{0,N+6} \right) \|\partial_3 X_p\|_{L_t^2(H^1)} + \left(\|\partial_3 Y_p\|_{L_t^2(L^2)} (1 + |(1 - S_p)\nabla Y_p|_{0,0}) \right. \\ &\quad \left. + |\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},0} \|\nabla Y_p\|_{0,0} \right) \|\partial_3 X_p\|_{L_t^2(H^{N+6})} + \left(|\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},1} (\|\nabla X_p\|_{0,N+6} (1 + \|\nabla Y_p\|_{0,0} \right. \\ &\quad \left. + |(1 - S_p)\nabla Y_p|_{0,0}) + \|\nabla Y_p\|_{0,N+6} (|\nabla X_p|_{0,0} + \|\nabla X_p\|_{0,0})) \right. \\ &\quad \left. + \|\nabla Y_p\|_{0,N+6} |(1 - S_p)\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},1} \|\nabla X_p\|_{0,1} \right) + \|\partial_3 Y_p\|_{L_t^2(H^3)} + \|\partial_3 Y_p\|_{L_t^2(H^{N+6})} \times \\ &\quad \times \left(\|\nabla Y_p\|_{0,0} |\partial_3 X_p|_{\frac{1}{2}+\bar{\varepsilon},0} + (|\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},0} + |(1 - S_p)\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},1}) \|\nabla X_p\|_{0,1} \right. \\ &\quad \left. + \|\partial_3 Y_p\|_{L_t^2(H^1)} |\nabla X_p|_{0,0} \right) + \|\nabla Y_p\|_{0,N+6} |\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},0} \|\nabla Y_p\|_{0,0} |\partial_3 X_p|_{\frac{1}{2}+\bar{\varepsilon},0} \\ &\quad + (|(1 - S_p)\nabla Y_p|_{0,0} \|\nabla X_p\|_{0,1} + \|\nabla Y_p\|_{0,1} \|\nabla X_p\|_{0,0}) \\ &\quad \times (|\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},1} \|\partial_3 Y_p\|_{L_t^2(H^{N+6})} + \|\nabla Y_p\|_{0,N+6} |\partial_3 Y_p|_{\frac{1}{2}+\bar{\varepsilon},1} \|\partial_3 Y_p\|_{L_t^2(H^3)}). \end{aligned}$$

Inserting (A.22) into the above inequality for $N = 0$ gives

$$\|e'_{p,1}\|_{L_t^1(0)} \lesssim \eta^2 (\theta_p^{-\beta+5\bar{\varepsilon}} + \theta_p^{-\gamma-\beta+5\bar{\varepsilon}} + \theta_p^{-\gamma+\bar{\varepsilon}}) \lesssim \eta^2 \theta_p^{-\gamma+5\bar{\varepsilon}}.$$

Whereas for $N \leq N_0 - 6$ such that $-\beta + \bar{\varepsilon}(N + 5) \geq \bar{\varepsilon}$, by substituting (A.24) into the above inequality, we achieve

$$\|e'_{p,1}\|_{L^1_t(\delta,N)} \lesssim \eta^2(\theta_p^{-\beta+\bar{\varepsilon}(N+5)} + \theta_p^{-\gamma-2\beta+\bar{\varepsilon}(N+7)} + \theta_p^{-\gamma-\beta+\bar{\varepsilon}(N+6)}) \lesssim \eta^2\theta_p^{-\beta+\bar{\varepsilon}(N+5)}.$$

Then (10.23) follows by interpolating the above two inequalities.

• The proof of (10.24)

Applying (9.14) to $e''_{p,0}$ gives,

$$\begin{aligned} \|\langle t \rangle^{\frac{1}{2}} e''_{p,0}\|_{L^2_t(\delta,N)} &\lesssim \|\nabla X_p\|_{0,0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L^2_t(H^{N+6})} + \|\nabla X_p\|_{0,N+6} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L^2_t(L^2)} \\ &+ \sum_{j=p}^{p+1} \left(|\partial_t Y_j|_{1+\bar{\varepsilon},1} \|\nabla X_p\|_{0,N+6} \|\nabla X_p\|_{0,0} + \|\nabla Y_j\|_{0,N+6} |\nabla X_p|_{0,0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L^2_t(L^2)} \right. \\ &\quad \left. + (\|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_j\|_{L^2_t(H^{N+6})} + \|\nabla Y_j\|_{0,N+6} |\partial_t Y_j|_{1+\bar{\varepsilon},1}) |\nabla X_p|_{0,0} \|\nabla X_p\|_{0,0} \right). \end{aligned}$$

Again due to $\beta \geq 7\bar{\varepsilon}$, we deduce from (10.10) that

$$(A.26) \quad \|\nabla Y_{p+1}\|_{0,6} + \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_{p+1}\|_{L^2_t(H^6)} \leq C\eta.$$

As a result, it comes out

$$\|\langle t \rangle^{\frac{1}{2}} e''_{p,0}\|_{L^2_t(\delta,0)} \lesssim \eta^2 \theta_p^{-\beta-\gamma+5\bar{\varepsilon}}.$$

The case when $N \leq N_0 - 6$ with $-\beta + \bar{\varepsilon}(N + 5) \geq \bar{\varepsilon}$, it follows from (10.10) that

$$(A.27) \quad \|\nabla Y_{p+1}\|_{0,N+6} + \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_{p+1}\|_{L^2_t(H^{N+6})} \leq C\eta \theta_p^{-\beta+\bar{\varepsilon}(N+5)},$$

so that in this case, we have

$$\|\langle t \rangle^{\frac{1}{2}} e''_{p,0}\|_{L^2_t(\delta,N)} \lesssim \eta^2 \theta_p^{-\beta-\gamma+\bar{\varepsilon}(N+5)}.$$

(10.24) follows by interpolating the above inequalities.

• The proof of (10.25)

Applying (9.14) to $e'_{p,0}$ gives

$$\begin{aligned} \|\langle t \rangle^{\frac{1}{2}} e'_{p,0}\|_{L^2_t(\delta,N)} &\lesssim \left(\|\nabla Y_p\|_{0,N+6} ((1 + |(1 - S_p) \nabla Y_p|_{0,0}) \|\nabla X_p\|_{0,0} + \|\nabla Y_p\|_{0,0} |\nabla X_p|_{0,0}) \right. \\ &\quad \left. + \|\nabla Y_p\|_{0,0} \|\nabla X_p\|_{0,N+6} \right) |\partial_t Y_p|_{1+\bar{\varepsilon},1} + \|\nabla Y_p\|_{0,0} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L^2_t(H^{N+6})} \\ &\quad + (1 + |(1 - S_p) \nabla Y_p|_{0,0}) \|\nabla Y_p\|_{0,N+6} \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t X_p\|_{L^2_t(L^2)} \\ &\quad + (\|\nabla X_p\|_{0,N+6} + \|\nabla Y_p\|_{0,N+6} |\nabla X_p|_{0,0}) \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_p\|_{L^2_t(L^2)} \\ &\quad + ((1 + |(1 - S_p) \nabla Y_p|_{0,0}) \|\nabla X_p\|_{0,0} + \|\nabla Y_p\|_{0,0} |\nabla X_p|_{0,0}) \|\langle t \rangle^{\frac{1}{2}} \nabla \partial_t Y_p\|_{L^2_t(H^{N+6})}. \end{aligned}$$

Then as in the proof of (10.24), we deduce that

$$\|\langle t \rangle^{\frac{1}{2}} e'_{p,0}\|_{L^2_t(\delta,0)} \lesssim \eta^2 \theta_p^{-\gamma+5\bar{\varepsilon}},$$

and

$$\|\langle t \rangle^{\frac{1}{2}} e'_{p,0}\|_{L^2_t(\delta,N)} \lesssim \eta^2 \theta_p^{-\beta+\bar{\varepsilon}(N+5)}.$$

for $N \leq N_0 - 6$ with $-\beta + \bar{\varepsilon}(N + 5) \geq \bar{\varepsilon}$. Then (10.25) follows by interpolating the above inequalities.

This ends the proof of Lemma 10.3.

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